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THE MATHEMATICS TEACHER

Volume XXV Number 3

Edited by William David Reeve

Mastery of Certain Mathematical Concepts by Pupils at the Junior High School Level

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University High School, Columbia, Missouri

Chapter I

INTRODUCTION

Purposes of the Investigation

The investigation which will be reported in the following pages was undertaken mainly for the purpose of finding out to what extent a certain group of mathematical concepts had been mastered by representative groups of pupils upon completion of the seventh, eighth, and ninth grades. In this investigation the attempt has been made to adduce evidence bearing upon three main questions:

1. To what extent has the group of mathematical concepts as a whole been mastered by pupils upon completion of the seventh, eighth, and ninth grades?

2. To what extent have the individual concepts and sub-groups of

concepts been mastered by pupils upon completion of the seventh, eighth and ninth grades?

3. What, if any, is the relation of mastery of these concepts to mental age and to chronological age?

Accordingly the investigation proper will be reported in three separate divisions, a chapter of the report being devoted to each of the three major problems which have been mentioned.

The Nature of the Investigation

The investigation is essentially a survey of prevailing conditions rather than a controlled experiment. The techniques employed are the simple statistical techniques involved in any study of averages and distributions of statistical data. The purpose has been merely to ascertain as carefully as possible the status of mastery of the concepts under ordinary present-day conditions. So far as possible care was taken to insure that the data were drawn from representative situations.

Justification of the Study

The important rôle of mathematical concepts in the development of mathematical ability can hardly be doubted. This would seem to be particularly true with respect to those abilities above the level of mere computation. The development of these higher abilities or, as Everett[1] calls them, "associative skills" which involve real insights into functional relationships, has come to be regarded as of primary importance in elementary mathematical instruction, and it would seem to imply mastery of the quantitative and spatial concepts among which these relationships exist.

The National Committee on Mathematical Requirements, in formulating and defining the aims of instruction in elementary mathematics, makes the following statement:[2]

> In formulating the disciplinary aims of the study of mathematics the following should be mentioned: (1) The acquisition, in precise form, of those *ideas or concepts* in terms of which the quantitative thinking of the world is done. . . . (2) The development of *ability to think clearly in terms of such ideas or concepts*. (3) . . . Training in "functional thinking," that is thinking in terms of and about relationships, is one of the most fundamental disciplinary aims of the teaching of mathematics. . . .

[1] Everett, J. P., *Fundamental Skills of Alg.*, pp. 12-13. Bureau of Publications, Teachers College, Columbia University.

[2] Young, J. W., et al., *Reorganization of Mathematics in Secondary Education*, p. 9.

Meltzer[3] also points out that concepts are of fundamental importance in determining reactions to situations. He says, in part " 'Makes conduct adaptable' is just what concepts do . . . our reactions are largely determined by the meanings which situations have for us. . . ."

In his recent study concerning the training in mathematics essential to success in college physics, Congdon[4] points out and illustrates the poor results attending failure to understand certain mathematical concepts involved in problem situations and emphasizes the importance of presenting the concepts adequately and clearly. In his summary chapter he says, in part (p. 89):

> The frequency of certain words in the mathematical vocabulary . . . suggests the advisability of additional emphasis in high school mathematics classes upon certain concepts. . . .

The following quotation is taken from the preface to Breslich's new ninth grade textbook:[5]

> . . . Ninth grade algebra stands out as one of the most difficult subjects of the modern curriculum. This is largely because of the way it is presented. Some of the major difficulties are: (1) The fundamental concepts are not sufficiently clear to the learner because they are presented abstractly and too rapidly in succession. Hence the pupil fails to acquire a full understanding of the meaning of such important concepts as literal number, signed number, exponent, equation, and relationship. . . . The fundamental concepts of algebra are given a concrete setting by correlating them with intuitive geometry.

These quotations and others in similar vein indicate that the development of ability to do functional thinking is one of the great aims of mathematical instruction, and that mastery of concepts is indispensable to the achievement of this aim. Therefore it would seem reasonable to suppose that not only would a great deal of instructional effort be directed toward the development of concepts, but also that there would have been accumulated a considerable amount of information regarding the actual degree to which the concepts are mastered at various instructional levels.

This, however, seems not to be the case. It is impossible, of course, to know how much has been done along this line by individual teachers working with their own classes, but it is certain that in the

[3] Meltzer, *Children's Social Concepts*, p. 5.

[4] Congdon, A. R., *Training in Mathematics Essential for College Success*, pp. 37-38 and 89. Bureau of Publications, Teachers College, Columbia University.

[5] Breslich, E. R., *Senior Mathematics, Bk. I*, p. xiii. University of Chicago Press.

domain of published tests the specific testing for mastery of mathematical concepts has received scant attention.

It is probably true that mathematical concepts are component elements in the development of most mathematical skills and in the performance of most mathematical acts and processes and thus they may have been accounted for indirectly in tests of such skills and of the ability to perform such processes. On the other hand, it is certainly true that there has been a noteworthy lack of purposeful effort to ascertain the degree of mastery of mathematical concepts *per se*. It is this fact which has given rise to the present investigation, and the justification of the study is to be found in the situation which has been described.

Fundamental Limitations and Assumptions

It must be obvious that a statistical study conducted by an individual without subsidy will inevitably be subject to certain limitations. Moreover, every study of this sort must be based upon certain fundamental assumptions. The present study is no exception to this, and all inferences drawn from the study should be viewed in the light of those limitations and assumptions. It will be well, therefore, to have them in mind throughout the report.

Limitations of the Study

(a) The list of concepts upon which the test was built is not original with the author of this report but was taken bodily from another investigation, that of Raleigh Schorling,[6] reported in 1925. No effort will be made to validate the selection of the concepts which make up this list. While it is admittedly incomplete, it is the basic list arrived at as the result of a very extensive scientific study, which commanded facilities of time, material, and co-operating personnel beyond the reach of the private student. Blackhurst[7] says of it:

> . . . The study by Raleigh Schorling is the most thorough and scientific attempt made to date to determine a specific list of objectives for Junior High School mathematics. . . .

Schorling devotes 52 pages of his report to a description and discussion of the five bases of selection and validation of the items in

[6] Schorling, R., *Objectives in Teaching of Junior High School Mathematics*, pp. 101-102. George Wahr.

[7] Blackhurst, J. H., *Prin. and Meth. of Junior High School Mathematics*, p. 82.

his lists. It is felt, therefore, that any attempt on the part of the present writer further to justify the composition of the list would be futile. The validity of the test items will be assumed therefore at the outset, and any attempt either to criticize or defend the selection will not be considered germane to the present discussion.

(b) A second limitation is the restriction of the investigation of mastery of these concepts mainly to that attained by pupils upon completion of the seventh, eighth, and ninth grades. Certain limited studies involving 379 cases of tenth, eleventh, and twelfth grade pupils and 281 cases of midyear pupils in the seventh, eighth, and ninth grades will be mentioned, but the main emphasis will be laid upon the status and development of mastery at the time of completion of the seventh, eighth, and ninth grades.

(c) While the results of this study may have certain psychological bearings it is not essentially a psychological study. No effort has been made to ascertain by what processes mastery of the concepts is attained. On the contrary, the purpose has been merely to determine how far mastery has advanced at various educational levels.

(d) A very important limitation that must be pointed out is that no concept will be studied in more than one setting or context. It is important to make this clear because, as Meltzer[8] points out, a concept is:

> . . . a bundle of possible "cues" to adjustment, the particular cue to be acted upon being determined by the situation or problem. . . .

For example, the concept "angle" has a somewhat different connotation when thought of in its variable or "functional" aspects from that which it has when thought of in its static aspect as an angle of a particular fixed size. There is to be considered, of course, the possibility that a pupil might have mastered a given concept in relation to a different setting or "cue" from that indicated in the test and still give an incorrect response to that concept as presented in the test. A consistent effort has been made to study each of the concepts with reference to what the author believed to be its most common context or setting. The author dares not claim infallibility on this point, but every one of the sixty-three concepts was considered carefully from this standpoint before being set into the test which served as the basic instrument for securing the data upon which this study is based.

[8] Meltzer, *Children's Social Concepts*, p. 2.

(e) The numbers of pupils responding correctly and incorrectly to a given concept will not be regarded at all as necessarily giving any indication of the inherent "difficulty of mastery" of that concept. It is quite possible that there are some concepts which are capable of being mastered rather easily but which might elicit incorrect responses from many pupils merely because these concepts are outside of the pupils' contact and experience. Of two concepts the one involving the lesser inherent difficulty of mastery might show a decidedly lower degree of mastery than the other.

(f) Finally there must be acknowledged the uncertainty as to the intrinsic difficulty involved in the statements of the various items in the test. There seemed no way to insure against possible inequalities of this nature, since there are no known indices of difficulty for the concepts themselves. The best that could be done was to try to present each test item in such a clear and simple fashion that there would be a minimum of likelihood of misunderstanding due to the statement or context itself.

The following assumptions are made in the belief that they are reasonable and that it is necessary to postulate them in order to make possible any significant interpretations of the results of the investigation.

Fundamental Assumptions

(a) It will be assumed that the list of sixty-three concepts used as a basis for this investigation is composed of items which really are mathematical concepts and which are valid objectives for junior high school mathematics. It will further be assumed that it is the best list at present available.

(b) It will be assumed that a correct response to an item in the test implies mastery of the concept involved, and that an incorrect response implies lack of mastery of that concept.

(c) It will be assumed that the pupils tested represent a typical cross-section of the seventh, eighth, and ninth grade school population.

(d) It will be assumed that each concept is presented in the test in its most common setting or context.

(e) It will be assumed that incorrect responses to test items are due to lack of mastery of the concepts involved and not to lack of clearness or simplicity in the statement of the test items.

(f) It will be assumed that the interpretation given to the concepts for the purpose of evolving test items is valid.

(g) It will be assumed that in the effort to bring the definition or illustration of a concept within the range of understanding of pupils of junior high school age, the mathematical correctness of the statement or illustration, as far as it goes, has suffered no violation, although in some cases rigorous completeness of definition has been sacrificed in the interest of clearness.[9]

Definition of Terms Used in This Study

For the purposes of this study the following terms will be used as defined below.

The Score Made by a Pupil on the Test: The total number of test items for which the pupil gave the correct responses.

Mastery of a Concept: The demonstrated ability to recognize or identify the definition, description, or illustration of a concept or the appropriate usage of the word or words naming the concept in the test used.

Coefficient of Mastery: The per cent of the given group who respond correctly to a given item in the test. This term is also applied to given groups of items and in such cases will mean a weighted arithmetic mean of the individual coefficients for the several items in the group.

"Set" of Coefficients of Mastery: The three mastery coefficients representing the mastery of a given item by a given grade in the (a) large, (b) medium, and (c) small schools.

Sources of Data

This study is based entirely upon original data secured by the investigator through the co-operation of administrative authorities and teachers in nine co-operating schools. The principal original data consist of the test booklets of 1658 pupils in the seventh, eighth, and ninth grades of these nine schools and of intelligence quotients for 279 of these pupils. The 1658 pupils include both girls and boys. The schools ranged in size from large metropolitan schools to small village schools, and included schools having junior high school organization as well as those organized on the traditional 8-4 plan. The pupils were

[9] For justification of this point of view see any of the following references: (a) Schaaf, W. L., *Mathematics for Junior High School Teachers*, pp. 28 and 36; Johnson Publishing Co.; (b) Schultze, Arthur, *Teaching of Math. in Sec. Sch.*, Chap. IV and V; Macmillan Co.; (c) Smith, D. E., and Reeve, W. D., *Teaching of Junior High School Mathematics*, p. 42; Ginn and Co.

of all ages from about eleven years to over sixteen years, and included pupils of very diverse intellectual capacities as indicated by the intelligence quotients which were secured. All possible care was taken to insure that the data be comparable and that it be representative of a typical situation in the large. A detailed description of the test used and of the details of securing and handling the data will be given in subsequent sections of this report.

The Construction of the Test

After the selection of the items to be included in the list of concepts, the next task was to devise an instrument for determining the mastery or lack of mastery of these items. It appeared that this measuring instrument would necessarily be in the nature of a test containing all of the items, and because of the large number of cases deemed necessary to insure results of substantial reliability it was felt that it must be capable of being administered as a group test.

The form selected was that of the multiple choice or recognition test with four responses provided for each of the sixty-three items. This having been selected as a working basis, the actual construction of the test, item by item, was begun. It was immediately apparent that different ones of the items lent themselves best to setting in the test in different forms. Certain geometric concepts, for example, are quite difficult of definition but can easily and clearly be represented by diagrams. Others (e.g., ratio) not essentially geometric in character, are difficult of definition but can be illustrated in such a way as to be identified with comparative ease. Still others (e.g., cylinder) were of such a nature that it was found desirable to provide for identification of the concept with an outstanding characteristic of some ordinary thing which is presumably within the experience of every child, and which embodies as one of its outstanding characteristics the thing or feature named in the concept.

It therefore became necessary to define at once what the term "mastery of a concept" should be taken to mean. After careful consideration had been given to this it was arbitrarily decided that for the purpose of this study mastery of a concept should be considered as having been attained if the pupil is able to recognize or identify an illustration, definition, description, or explanation of the concept, or to identify an essential characteristic of the concept with something which fundamentally embodies this defining characteristic. It was further assumed that the ability properly to associate the concept with

the illustration, definition, etc., and the ability properly to associate the definition, illustration, etc., with the concept should be equally regarded as evidence of mastery of the concept.[10] These assumptions or definitions allowed greater freedom in the actual construction and statement of the test items, while in the opinion of the author they detracted nothing from the essential meaningfulness (in the light of limitation [d] page 121) of the items.

The thought underlying the whole test was that each concept should be presented in the form in which it appeared to the author most likely that a pupil would be able to recognize or identify it, whether by description, definition, illustration, or association with some commonplace thing. The aim was not to make the test difficult. Rather, the fundamental assumption was made that if a pupil could recognize or identify a concept in the simplest way in which it could be presented to him as a general concept he should be presumed to have attained mastery of that concept, and the responses to the test items were selected from this point of view. The order in which the concepts appear in the test was determined by pure chance, as was the position of the correct response among the four responses provided for each item.

Establishing the Reliability of the Test

In the spring of 1929 a mimeographed form of the whole test (including the heading, preliminary instructions to the pupil, one example illustrating the method of indicating the correct response, and the sixty-three four-response items of the test itself), was prepared. During the summer session of 1929 this form of the test was given to 49 pupils ranging from grade seven to grade twelve, and to 11 third- or fourth-year college students. One of the papers had to be thrown out because one page was missing, but the other 59 papers were scored and tabulated and a preliminary study of the results was made.

It was found that for pupils in grades seven to twelve inclusive the total raw scores ranged from 16 to 55 on the basis of a possible 63, with a general progression from grade to grade. The total raw scores of the college students ranged from 58 to 62 inclusive.

When checked for reliability by correlating totals of correct responses on odd-numbered items with totals of correct responses on

[10] This point of view is expressed by Smith and Reeve in their book, ***Teaching of Junior High School Mathematics***, p. 42.

even-numbered items (using the product-moment formula) and then stepping up the coefficient thus obtained by means of the Spearman-Brown formula,[11] a coefficient of reliability of $.95 \pm .01$ for the whole test was obtained.

A further determination of the reliability of the test was made when on December 5, 1929, the test was administered to practically the entire student body of the University High School. This group, as did the first group, included pupils in all grades from seven to twelve inclusive. This second administration of the test yielded results substantially similar to those previously obtained with the smaller group. When both groups were thrown together the aggregate number of cases was 190. This time, using all 190 cases, the computation yielded a reliability coefficient of $.95 \pm .00$ for the whole test.

In this group of 190 cases there were 88 junior high school pupils. A fourth computation of the reliability coefficient was made, this time using only the 88 cases of junior high school pupils. It was found that the reliability coefficient of the whole test for these 88 cases of junior high school pupils alone was $.91 \pm .01$.

Because of the consistently high coefficients of reliability and the observation of a rather consistent average improvement from grade to grade, it was felt that it would be unwise to attempt any revision of the test. Another reason for this feeling was that the scores of the college students, all mathematically trained, were all very high. This was taken to indicate that from a mathematical standpoint and from the standpoint of accuracy of presentation of the test items the construction of the test was relatively satisfactory.

That the test is amenable to objective scoring is true without a doubt. All scoring was done by means of keys prepared by the author. A preliminary check was made on this by having 130 papers scored by a student teacher, and then every tenth paper in the group re-scored and checked by the author. Among the 819 items thus checked, only one error was found. Subsequent verification was made by having over 500 papers scored by two student teachers and seven senior students in college. Every one of these papers was re-scored and checked by the author and one assistant. Among the approximately 32,000 items scored by nine different and all inexperienced people only two errors in scoring were discovered while one error in counting up the number of correct responses on a paper was found.

[11] See Ruch and Stoddard, *Tests and Measurement in H. S. Instruction*, p. 358.

It is believed that these data indicate as high a degree of objectivity as could reasonably be expected of any test.

Conclusion

In view of the foregoing considerations the test was presumed to be a reasonably adequate measuring instrument for the determination of mastery of these sixty-three mathematical concepts. Therefore no changes were made in it, and this original form was printed as the final form of the test in February, 1930. A copy of the test follows.

BUTLER TEST FOR MATHEMATICAL CONCEPTS

—JUNIOR HIGH SCHOOL—

Prepared by C. H. Butler, Supervising Teacher of Mathematics,
University High School, University of Missouri

(SECOND EDITION)

Name.................................Boy or Girl.............

Grade........School.....................Date...................

City..................................State...................

I was........years old on the........day of............., 19......

Directions: For each of the following statements or questions four choices are given. These are numbered (1), (2), (3), and (4). One of the four completes the statement or answers the question more correctly than any of the others. Find the correct one in each case and *write its number* in the space at the right-hand edge of the page. Read the following example:

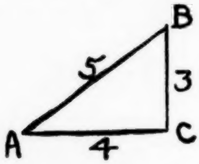
Example: The May reading of Mr. Jones' gas meter was 15,300 cu. ft. and the June reading was 15,600 cu. ft. The amount of gas used between the two readings was (1) 15,600 cu. ft.; (2) 300 cu. ft.; (3) 15,300 cu ft.; (4) 600 cu. ft. ....(2)

The number 2 was put in the parentheses because the second answer (300 cu. ft.) was correct. Now do the rest of the examples the same way.

---

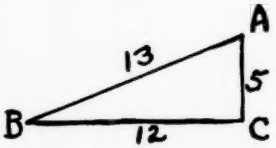
1. Which of the following expressions represents the ratio of two numbers? (1) $4-3$; (2) $6+7$; (3) $\frac{5}{2}$; (4) 8×5......................(3)

2. Which of the following can be found only by indirect measurement? (1) the distance around a wagon wheel; (2) the weight of an iron ball; (3) the voltage of an electric current; (4) the distance from the earth to the moon ....( )
3. In this figure the tangent of angle A is:



(1) $\frac{5}{3}$; (2) $\frac{3}{4}$; (3) 20; (4) 3 ....( )

4. Similiarity (in its mathematical meaning) means: (1) a method used by bankers to find how much money is due on a loan; (2) solving a problem by a particular rule; (3) having the same shape; (4) being alike in every way ....( )
5. A measurement is: (1) a ruler or yardstick; (2) the study of electric meters, water meters, etc.; (3) calculation of a distance by figuring it out; (4) finding out how many units in a certain amount. ....( )
6. The volume of a solid means: (1) how wide it is; (2) how many cubic units it contains; (3) its position; (4) its shape ....( )
7. Which of the following is most nearly like a circle? (1) a half-dollar; (2) ⅞; (3) a shoe box; (4) a baseball ....( )
8. Which of the following objects is shaped like a rectangular prism? (1) a ball; (2) a brick; (3) a tomato can; (4) a triangular sheet of paper ....( )
9. A root of an equation is (1) a specified sum of money mentioned in the problem; (2) an answer that is not correct; (3) a value of the unknown quantity which makes the equation true; (4) a statement that two of the numbers are equal ....( )
10. The four items listed below are about a coal pile, an airplane, a pencil, and a tree. Two properties of each are given. Read them all over and and then decide in which case one of the properties depends upon the other: (1) the size of a coal pile and the number of tons it contains; (2) the number of an airplane and the speed at which it can travel; (3) the color of a pencil and its cost; (4) the age of a tree and the kind of tree it is ....( )
11. In this figure the sine of angle A is:



(1) $\frac{5}{13}$; (2) Angle C; (3) $\frac{12}{13}$;
(4) $5 + 12 + 13$ ....( )

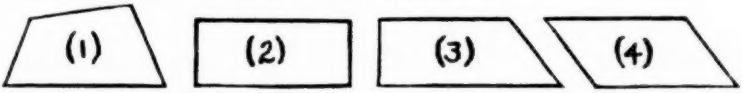
12. Which has the greater number of surfaces? (1) a marble; (2) a sheet of paper in the shape of a triangle; (3) a brick; (4) a long piece of wire ....( )

13. Which of the following is a formula? (1) $A = \frac{1}{2}b \times h$;

(2)  (3) $11 = 5 + 6$; (4) 43% .... ( )

14. Pi (sometimes written π) means: (1) the name of an Italian coin; (2) the number of times the diameter of a circle can be divided into the circumference; (3) the amount which has to be paid annually on an insurance policy; (4) an algebraic number that can have different values according to the conditions of the problem in which it is used .... ( )

15. Which of the following figures is a trapezoid?



.... ( )

16. The average of any seven numbers is: (1) the middle number of the seven; (2) a number which is the sum of the seven numbers we started with; (3) a number which is obtained by adding all seven numbers together and dividing by seven; (4) the number which is obtained by multiplying all seven numbers together and dividing by seven .... ( )

17. Which of the following represents a pyramid? (1) a flat geometric figure with five straight sides; (2) 

(3) $7 + 3x = 13$; (4)  .... ( )

18. A cone is: (1) one of the four chief methods of showing how numbers or quantities are related; (2) a geometrical object that has a square base and comes up to a point; (3) a word statement of a problem; (4) an object shaped about like the well-sharpened end of a pencil .. ( )

19. Which of the following numbers is the fourth power of 3 ? (1) 12;

(2) 64; (3) 81; (4) $\frac{4}{3}$ .... ( )

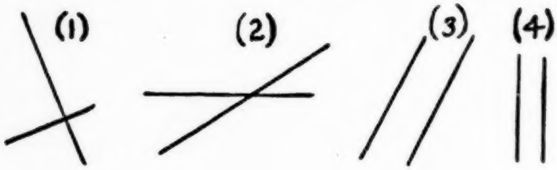
20. If I buy a book for \$2.00 and sell it for \$1.85, the 15c difference represents my (1) investment; (2) interest; (3) loss; (4) profit .... ( )

21. In the expression $4x^2 + 17y$, the coefficient of y is: (1) 3; (2) $17y$; (3) $4x^2$; (4) 17 .... ( )

22. A statistical graph is: (1) a display of numerical facts by means of a sort of picture; (2) a method of calculating the distance between two points; (3) an arrangement of numbers in rows and columns; (4) a design made from geometrical figures .... ( )

23. The simple interest on \$100.00 for two years at 5% is: (1) \$5.00; (2) \$4.00; (3) \$10.00; (4) \$2.00 .... ( )

24. Taxation is a method of: (1) learning more about money; (2) raising money for public purposes; (3) getting immediate payment for goods sold; (4) figuring up the total expenses of a business .... (   )
25. A solution of an equation means: (1) the correct answer; (2) a formula; (3) two numbers which must be multiplied together; (4) the problem from which the equation comes .... (   )
26. A baseball has the general shape of: (1) a circle; (2) a quadrilateral; a sphere; (4) a perimeter .... (   )
27. Which of the following reasons best explains why algebra has been called "a tool of science"? (1) algebra is a science itself; (2) most people who study algebra become interested in science; (3) the use of algebra makes it easier to state scientific laws and to work with them; (4) science is taught in laboratories, while algebra is not ...... (   )
28. Read this problem carefully: "Find the distance covered in any given number of hours by a train moving at 40 miles per hour." This is a problem involving: (1) quadratic equations; (2) equivalent fractions; (3) direct variation; (4) metric measurement .... (   )
29. In the expression $7^2 - 9 = 40$, the exponent of 7 is: (1) 2; (2) 9; (3) 40; (4) 49 .... (   )
30. The thing we use to show numerical facts by means of a picture is called: (1) a table of statistics; (2) a graph; (3) a logarithm; (4) an equation .... (   )
31. Which of the following pairs of lines are perpendicular to each other?



.... (   )

32. Which of the following expressions is a proportion?

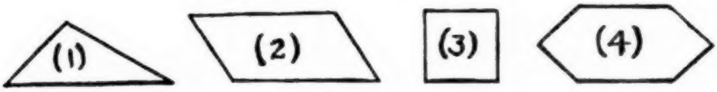
(1) 51%; (2) $\frac{3}{4} = \frac{9}{12}$; (3) $7 \times 82 =$; (4) $13 + 6 - 2$.............. (   )

33. Insurance is: (1) financial protection against loss; (2) a business organization; (3) the payment of a certain sum of money every month or year; (4) loss of money by fire or accident .... (   )
34. If I buy a book for $1.00 and sell it for $1.35, my profit is (1) $1.00; (2) $1.35; (3) $2.35; (4) 35c .... (   )
35. Congruence means; (1) the comparison of measurements in the metric and the English systems; (2) the correctness of an estimate (for the answer to a problem); (3) a comparison of the sizes of two angles; (4) being exactly alike (in the case of two or more geometric figures)..(   )
36. Which of the following expressions is an equation? (1) 4×3; (2) $18 - 6 + 4 + 11 - 7$; (3) $42\frac{1}{2} \div 4$; (4) $6 + 11 = 17$ .... (   )

37. If positive numbers represent miles of travel eastward, then negative numbers represent: (1) hours spent in travel eastward; (2) miles of travel southward; (3) miles of travel westward; (4) miles of travel northward .... (3)
38. If I loan \$100 at 6% compound interest for ten years, the amount which will draw interest the second year is: (1) \$106; (2) \$100; (3) \$6; (4) \$94 .... (1)
39. Which of the following expressions is an algebraic fraction? (1)

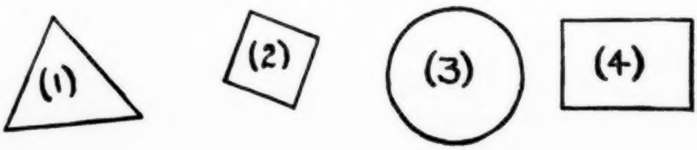
(1) $\frac{5}{3}$ (2) $\sqrt{\frac{3}{7}}$ (3) $\frac{2a}{3}$ (4) $\frac{15}{12} \div \frac{6}{7}$ .... (3)

40. Which of the following objects is shaped like a cylinder? (1) a ball; (2) a brick; (3) a tomato can; (4) an automobile tire .... (3)
41. Which of the following figures is a triangle?

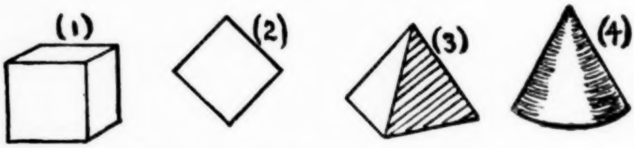


.... (1)

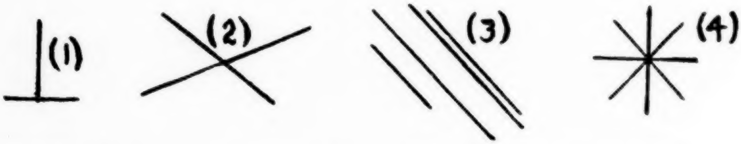
42. Graphic representation means: (1) dishonest business methods; (2) investing money so that it will draw interest; (3) showing mathematical facts by means of pictures; (4) working problems by means of calculating machines .... (3)
43. Sometimes letters are used for numbers, and sometimes letters and numbers have different values according as they have plus signs or minus signs in front of them. We call such letters and numbers: (1) integers; (2) algebraic numbers; (3) positive numbers; (4) geometrical numbers .... (2)
44. The name of this figure  is: (1) rectangle; (2) triangle; (3) perpendicular; (4) parallelogram .... (4)
45. An algebraic product is: (1) the result we get when two or more algebraic numbers are multiplied together; (2) the general method of solving an equation; (3) the answer we get when we add one algebraic number to another; (4) one algebraic number subtracted from another .... (1)
46. Which of the following figures is a square? .... (2)



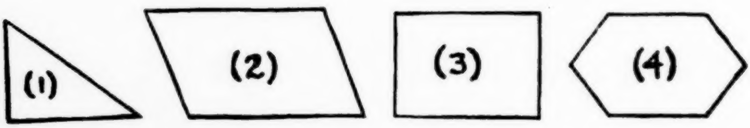
47. Which of the following figures looks most like a cube? ....( )



48. Which of the following sets of lines are parallel lines? ....( )



49. An algebraic factor is: (1) any algebraic number; (2) two or more algebraic numbers multiplied together; (3) an algebraic number added to some other number; (4) an algebraic number which is multiplied by some other number ....( )
50. The length of an object might be measured in: (1) pounds; (2) kilowatts; (3) seconds; (4) inches ....( )
51. The mathematical meaning of equality is: (1) the correct answer for a problem; (2) two or more things having the same value or containing the same number of units of the same kind; (3) a system of measurement used by the British and American people; (4) a problem which may be solved by either of two different methods ....( )
52. The fifth root of 32 is: (1) $\frac{32}{5}$; (2) 5×32; (3) 8; (4) 2 ....( )
53. Which of the following figures is a rectangle? ....( )

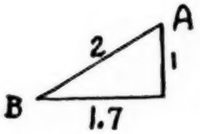


54. An algebraic expression containing the second power of the unknown quantity is called: (1) a linear equation; (2) a quadratic; (3) a radical (4) a logarithm ....( )
55. A map of your town would be: (1) a statistical graph; (2) a drawing to scale; (3) a perspective drawing; (4) a three-dimensional picture..( )
56. The distance around a flat figure is called its: (1) area; (2) perimeter; (3) cross section; (4) diameter ....( )
57. The following list contains three examples of measurement that are familiar to most people. These are: the length of a radio wave, the weight of a load of coal, and the length of a minute as determined by

the United States Naval Observatory instruments. ***HOW MANY*** of these measurements are absolutely exact (that is, how many involve no error whatever)? (1) all three; (2) two; (3) one; (4) none ...... ( )

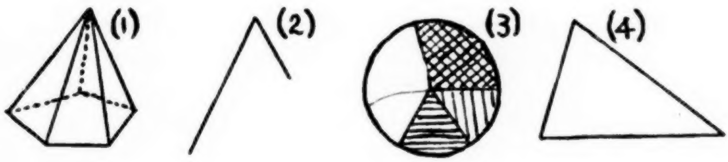
58. A merchant buys an article for \$6.00. He marks it to sell for \$8.00 but agrees to reduce this to \$7.50 if the purchaser pays cash. This amount (\$7.50) is known as: (1) the list price; (2) the net price; (3) the discount; (4) the rate of exchange .... ( )

59. Which of the following is a measure of area? (1) square yard; (2) gallon; (3) foot; (4) cubic inch .... ( )

60. 

In this figure the cosine of angle A is: (1) angle B; (2) ½; (3) a right angle; (4) 1.7 × 2 .... ( )

61. Which of the following is an angle? .... ( )



62. A point where two or more lines meet is called: (1) an axis; (2) a base; (3) a vertex; (4) a side .... (3)

63. An "approximation in a measurement" means: (1) the thing that is being measured; (2) the instrument that is used to make the measurement; (3) applying the result of the measurement in working out a problem; (4) a result that is not exactly correct .... (4)

The Co-operating Schools and the Representative Character of the Group of Pupils Who Took the Tests

In the spring of 1930 the co-operation of a number of schools was solicited in order to make it possible to have the tests administered to a large number of pupils in the seventh, eighth, and ninth grades. It was desired to have the tests given in schools of different types so that if any significant differences existed they could be noted. Therefore the list of schools originally selected contained schools in communities of significantly different sizes: some in small rural communities of less than 2,500 population, some in small cities between 2,500 and 25,000 in population, and some in large cities of over 25,000 population. These three classifications of schools will hereafter be referred to as the small schools, the medium schools, and the

large schools, or as the S-, M-, and L-groups of schools. The selection of the schools was made with the deliberate purposes of keeping a fair balance of total numbers of cases among the three groups of schools and of including some schools organized as junior high schools as well as some organized on the traditional 8-4 basis. Almost no difficulty was encountered in securing the co-operation of teachers and administrative officers of the schools.

It was found that with the exception of one school it was not feasible to give the tests to all of the seventh, eighth, and ninth grade pupils in the medium and large schools, and so the request was made that so far as possible representative sections in each grade be selected.

Not all of the tests which were sent out were actually administered and returned. Some of the schools did not use quite all the tests sent to them, and two schools failed to return any tests at all. These were both small schools, and their failure to return the tests makes the total number of cases in this group considerably less than was anticipated and smaller than the totals for the other two groups. In the opinion of the author, however, it is still large enough to give substantially representative results, particularly since it includes practically all pupils in grades seven, eight, and nine in five different small school systems. A total of 1,658 pupils in the seventh, eighth, and ninth grades were tested and their test papers were all returned to the author for scoring and treatment of the results.

In addition to these 1,658 tests which were all for grades seven, eight, and nine, the author had access to the papers of 379 senior high school pupils who took the same test. For the main part of the study, however, 1,377 papers were available. The remaining 281 of the 1,658 papers were those of pupils at the middle of grades seven, eight, and nine.

Administration of the Tests

In order to get comparable data it was thought necessary to have all the tests administered about the same time. Moreover it was desired that they be given as near the close of the school year as possible. Since some of the small schools closed by the middle of May it was necessary that they be given by that time. Accordingly arrangements were made to have all of the tests administered in all of the schools some time during the first two weeks of May, 1930. This arrangement was carried out as requested.

The tests were all administered by the regular classroom teachers, supervisors, or principals. Simple instruction sheets were sent to all the co-operating schools, and it was requested that the persons giving the tests do so in strict accordance with these instructions. A time limit of thirty minutes was imposed mainly for the sake of uniformity and for the prevention of excessive waste of time. Experience in the preliminary trials of the test had indicated that this allowance was ample for practically all pupils. A copy of the instructions is given here.

BUTLER TEST FOR MATHEMATICAL CONCEPTS
JUNIOR HIGH SCHOOL

(Preliminary Instructions to be Read to the Pupils by the Tester)

Fill in ALL the blanks at the top of the first page of your test booklet.

(Wait until all pupils have done this, and then read the following instructions:)

In the test which you are about to take, you *may* find some places where you do not know which of the four answers is the correct one or the best one. In such a case simply omit that item. *Do not guess at the answer.* The chances are three to one against you if you guess, and there is nothing to be gained by guessing. Answer only those items that you know. *Do not guess.*

Now read the directions on the first page while I read them aloud.

(The tester will now read aloud the "Directions" and the "Example" on the first page of the test booklet, and will see that each pupil understands what he is to do.)

No questions are to be asked or answered after the test has started. Observe this strictly.

Allow thirty minutes for the test.

Scoring

Of the 1,658 test papers which were administered in the nine schools, between 500 and 600 were scored by two student teachers and seven college students. The other papers were scored by the author and one assistant. After this was done the scoring of every one of the 1,658 papers was completely checked for accuracy. All the checking was done by the author and one assistant. Including the rescoring for checking, a total of 210,924 items were scored.

The scoring and checking was all done by means of keys prepared by the author, and the checking revealed that the amount of error in the original scoring was practically negligible.

Chapter II

GENERAL EXTENT OF MASTERY OF THE WHOLE LIST OF CONCEPTS

The performance of the various groups and subgroups of pupils was studied by means of group medians. Separate distributions of the scores of boys and girls in each grade of each school were first made and were then combined into whole-grade distributions for each school. The various combinations of these which were subsequently made yielded both grade-sex distributions and general grade distributions for each of the three classifications of schools and also for all the schools together. Altogether 141 distributions were made, and the median was calculated for each of these.

Summary of the Medians

It will be desirable first to examine the table of medians to see what differences exist among the large, medium, and small schools, either in the absolute median scores or in the grade-to-grade improvement as shown by the medians.

For convenience, the grade groups will be considered as wholes in the three classifications of schools separately and for all schools together. Only those pupils who were completing the grades in which they were enrolled at the time the test was given will be considered at this time. The medians are summarized in Table I which follows.

Table I

Summary of Seventh, Eighth, and Ninth Grade Medians for the Three Classifications of Schools and for All Schools Together

Classification	Medians		
	Grade 7	Grade 8	Grade 9
S-Group	25.93 ± .56	33.75 ± .65	39.95 ± .57
M-Group	30.83 ± .63	33.33 ± .46	46.25 ± .61
L-Group	25.58 ± .44	30.58 ± .50	38.00 ± .70
All Schools	26.91 ± .30	32.54 ± .31	40.50 ± .43

Examination of the above figures shows that no two of the classifications of schools have exactly the same median in any of the three grades. Some of the differences, however, are less than one score

point, and the largest difference is less than one-seventh of the entire scale.

The significance of the difference between two medians may be evaluated by comparing the obtained difference (D) with the standard error of the difference ($\sigma_{diff.}$) Garrett[12] points out that if the ratio $D/\sigma_{diff.}$ has a value equal to or greater than 3.00, there is virtual certainty that a true difference greater than zero exists. The significance of the differences between the grade medians in the three separate classifications of schools and the general grade medians was studied by evaluating the ratio $D/\sigma_{diff.}$ in each case, and in this way finding the probability of the existence of a true difference greater than zero. A summary of the results of these computations is given in Table II.

TABLE II

Probability of True Differences Greater Than Zero Existing Between Indicated Grade Medians in the Three Classifications of Schools and the Corresponding General Grade Medians

Classification	Grade	$D/\sigma_{diff.}$	No. of Chances in 100 that a True Difference Exists
Small Schools	7	1.00	84
Small Schools	8	1.13	87
Small Schools	9	.52	70
Medium Schools	7	3.70	100
Medium Schools	8	.96	83
Medium Schools	9	5.18	100
Large Schools	7	1.62	94
Large Schools	8	2.25	99
Large Schools	9	2.07	98

Table reads: For small schools, grade 7, $D/\sigma_{diff.} = 1.00$, indicating that there are 84 chances in 100 that the true median for the seventh grade of small schools differs from the true general seventh grade median by an amount greater than zero.

The data in the last column of Table II indicate that there is a considerable probability of a true difference between the general grade medians and the corresponding grade medians in the three classifications of schools. In only two of the nine cases, however, is the ratio $D/\sigma_{diff.}$ large enough to indicate a certainty of a true difference.

Each of the three groups of schools shows a consistent grade-to-grade improvement in mastery although the improvement is not entirely uniform in any of the three cases. The same thing is true of the general grade-to-grade improvement as shown by the general

[12] Garrett, H. E., *Statistics in Psychology and Education*, p. 136. Longmans.

grade-medians of the entire group of schools. The general improvement of the ninth grade over the eighth grade is greater than the improvement of the eighth grade over the seventh grade. However, the difference between the general ninth grade median and the general seventh grade median is less than one-fourth of the possible difference, and the general line of grade medians approximates fairly well a "straight-line" improvement. It is worth while to note that the average pupil, upon completion of the ninth grade, has mastered less than two-thirds of the concepts in the list. The medians of the large schools are lower in all three grades than the corresponding medians of either of the other groups of schools, and are, of course, lower than the corresponding general grade medians.

Achievement of the Mid-grade Groups

The mid-grade groups were composed of pupils who entered upon the work of their respective grades at the middle of the year and who, at the time the test was given, were completing the first half of their respective grades.[13] For convenience the data on these groups are summarized in Table III.

TABLE III
Medians of Mid-grade Groups

Grade	Pupils completing first half of		
	Grade 7	Grade 8	Grade 9
Number of Cases.......	79	103	99
Median...............	20.90 ± .77	30.17 ± .74	37.68 ± .58

These groups include pupils from only two schools and are smaller than the groups previously considered. Their medians, however, taken at their face value, seem to run fairly well in line with the other medians. On the whole these mid-grade medians tend to confirm the hypothesis of gradual and fairly uniform increase of mastery of the concepts up to the end of the ninth grade.

Mastery of the Concepts Above the Junior High School Level

In addition to the 1,658 cases used in the main body of this study, the investigator had access to scores made by 379 pupils in high

[13] In subsequent chapters grade groups will include only pupils completing the designated grades at the time the test was given, unless otherwise specified.

school physics classes on the Butler Test For Mathematical Concepts. A brief study of these scores will make possible certain tentative observations regarding the progressive mastery of the concepts beyond the junior high school level.[14]

The scores of the physics pupils were obtained from 14 different schools. In four of these the test was administered in May, 1930, and in each of the four cases the date of administration was within two weeks of the period during which the present investigator's tests were given. Accordingly the scores from these four schools have been regarded as "end-of-the-year" scores, just as were the 1,658 scores used in the main part of this study, and have been treated separately.

In the other ten high schools the tests were given at various times between November 7, 1930, and February 5, 1931. The numbers of cases in the several schools were judged to be too small to justify individual treatment. On the other hand there was a difficulty in treating them all together since they were not all given at the same time. However, the employment of a simple weighting device made it possible to calculate the average time of administration of the test to any particular group of pupils. The data concerning the mastery of the concepts by senior high school pupils are shown in detail in Table XX in the Appendix, and are here summarized in Table IV.

With these data it was possible to continue the study of improve-

TABLE IV

Description and Achievement of Senior High School Groups

Average grade status of group at time test was given	No. of pupils	No. of schools	Median
Completed 2.48 months of grade 11	91	10	$47.83 \pm .69$
End of grade 11	97	4	$51.50 \pm .61$
Completed 2.64 months of grade 12	172	10	$47.86 \pm .53$
End of grade 12	19	4	53.13 ± 1.13

Table reads: The first group had completed 2.48 months of eleventh grade work. The group includes 91 pupils in 10 schools. The median score for this group is $47.83 \pm .69$.

[14] The data for this part of the study were furnished by Dr. W. R. Carter, who, in connection with another investigation, administered the Butler Test for Mathematical Concepts to several hundred pupils in high school physics classes. Dr. Carter's treatment of the data is, however, different from that employed in this study. The present author wishes to acknowledge his obligation to Dr. Carter for his generous permission to use this material.

ment in mastery of the mathematical concepts through the senior high school period. There is one gap in the picture, since there were almost no scores for tenth grade students. The complete record of improvement in mastery of the concepts, as indicated by the general medians, is shown in Chart I.

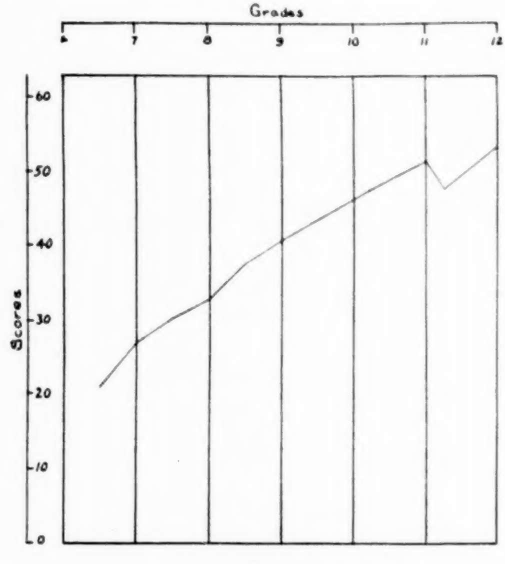


CHART I

Progressive Mastery of the Sixty-Three Mathematical Concepts as shown by the General Grade Medians from the Middle of Grade Seven to the End of Grade Twelve.

There is some reason to believe that the senior high school pupils used in the study in question might be expected to obtain higher scores than would be made by a perfectly representative group of pupils at the same grade levels. Intelligence tests and other tests indicated that a degree of selectivity obtains. Dr. W. R. Carter[15] has

[15] This statement was made by Dr. Carter during a personal interview with the author February 12, 1931.

some evidence which leads him to believe that the eleventh and twelfth grade medians may be as much as five or six score-points higher than would be found in large unselected groups of eleventh and twelfth grade pupils. So far as this group represents the true situation, however, there appears to be a consistent and steady improvement up to about the end of the eleventh grade. What happens beyond this point is problematic. A sharp drop in the line occurs between the end of grade eleven and the end of the first quarter of grade twelve, after which improvement apparently proceeds at about the same rate as in preceding grades, but in the end reaches a level very little above that attained at the end of grade eleven. The median for the end of grade twelve is based on only 19 cases which number is too small to warrant one in placing much reliance on the median, although this median appears to be consistent with the general trend of the preceding ones. One can only conjecture concerning the break in the line following the end of grade eleven. It is possible that similar breaks would appear after the completion of other grades if quarterly medians were plotted. There is reason to feel that the drop is a real one since the median in question is based on 172 cases in 10 schools.

At all events it appears that so far as these data are representative of the real situation, there is a general improvement throughout the senior high school period, or at least up to and through the eleventh grade.

Comparison of Grade-to-Grade Gains of Boys and Girls

The general grade-sex medians are summarized in Table V.

TABLE V

Medians for Boys and Girls, All Schools Together

	Grade 7	Grade 8	Grade 9
Girls	27.36 ± .43	31.90 ± .54	40.08 ± .51
Boys	26.64 ± .51	32.88 ± .41	41.36 ± .56
Chances in 100 of a true diff.	77	83	86

Last row of table reads: In grade 7 there are 77 chances in 100 that a true difference greater than zero exists between medians for boys and girls, etc.

In the whole group the seventh grade median for boys is found to be .72 score point below that for girls, but by the end of the eighth

grade the median for boys is .98 score-point higher than that for girls, and this lead is slightly increased by the end of the ninth grade. The probabilities of the existence of true differences between the median performances of boys and girls on the test are indicated in the last row of Table V. It appears that in each of the three grades the observed difference between the medians probably represents a true difference greater than zero, though the ratios $D/\sigma_{diff.}$ are not high enough in any of the three cases to give absolute assurance of this.

Table VI shows the gains made by boys, by girls, and by both together in each of the grade-to-grade intervals indicated. These were originally computed from the group medians. As a check upon these figures similar computations were made using the arithmetic means of the distributions instead of the medians. It can be seen that while the results are not identical they differ, with one exception, by amounts less than one score-point.

TABLE VI

Comparative Tables of Gains Computed from the Medians and from the Means of the Distributions

Average Gains in Score-Points	Computed from Medians			Computed from Means		
	Girls	Boys	Both	Girls	Boys	Both
Gain of grade 8 over grade 7............	4.54	6.24	5.63	4.39	6.61	5.42
Gain of grade 9 over grade 8............	8.18	8.48	7.96	7.05	7.56	7.34

Table reads: Based on computations from medians, eighth grade girls gain 4.54 score-points over seventh grade girls; eighth grade boys gain 6.24 score-points over seventh grade boys; and the eighth grade as a whole gains 5.63 score-points over the seventh grade. The remainder of the table should be read in a similar way.

It is apparent that whether the mean or the median is used as a measure of central tendency, the boys consistently master the concepts more rapidly than the girls do. The differences do not seem very large in comparison with the total number of concepts in the list, but when projected against the actual interval gains they become considerably more noticeable. The gains are all statistically significant since in each case the obtained difference or the gain is more than 3 times the $\sigma_{diff.}$. The superiority of the boys is relatively more manifest in the seven-eight interval than in the eight-nine interval.

The Spread of the Scores in the Various Groups

The groups medians, of course, are merely central tendencies and do not give complete descriptions of the performance in the groups of pupils who took the test. For the purpose of supplementing the picture which they afford it will be necessary to examine the dispersion of the scores in connection with the central tendencies. The data relative to this will be found in Tables VII, VIII and IX below.

TABLE VII

Summary of Data Relative to the Distributions of Seventh Grade Scores on the Butler Test for Mathematical Concepts

	Girls	Boys	Both
Range of Scores	1 - 46	3 - 46	1 - 46
Q_1-Q_3 Range	21.75 - 32.65	19.75 - 32.89	20.19 - 32.75
Number of Cases	261	236	497
Mean	27.00 ± .34	26.16 ± .41	26.60 ± .24
Median	27.36 ± .43	26.64 ± .51	26.91 ± .30
σ_{dis}	8.31	9.22	8.77
Skewness	−.130	−.156	−.106

Table reads: The range of scores for seventh grade girls is 1–46, for boys 3–46 . . . etc.

TABLE VIII

Summary of Data Relative to the Distributions of Eighth Grade Scores on the Test

	Girls	Boys	Both
Range of Scores	11 - 50	13 - 53	11 - 53
Q_1-Q_3 Range	25.39 - 37.35	27.40 - 37.20	27.37 - 37.76
Number of Cases	235	260	495
Mean	31.39 ± .43	32.77 ± .33	32.02 ± .24
Median	31.90 ± .54	32.88 ± .41	32.54 ± .31
σ_{dis}	9.86	7.84	8.16
Skewness	−.155	−.042	−.191

Table VIII is to be read in the same way as Table VII.

TABLE IX

Summary of Data Relative to the Distributions of Ninth Grade Scores on the Test

	Girls	Boys	Both
Range of Scores	5 - 55	12 - 58	5 - 58
Q_1-Q_3 Range	31.72 - 45.65	34.09 - 46.66	33.02 - 46.11
Number of Cases	198	187	385
Mean	38.44 ± .41	40.33 ± .45	39.36 ± .34
Median	40.08 ± .51	41.36 ± .56	40.50 ± .43
σ_{dis}	8.51	9.04	10.03
Skewness	−.578	−.342	−.341

Table IX is to be read in the same way as Table VIII.

At this point it will be sufficient to consider merely the three composite end-of-the-grade distributions for all schools together. The distributions are shown graphically in Chart II. Examination of Chart II reveals an extremely wide range covered by the scores in each of the three grades, with a concomitant large amount of overlapping of the three distributions.

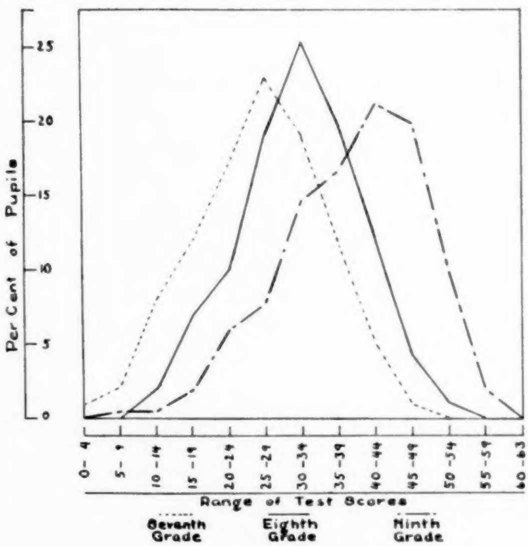


CHART II

Distributions of Scores on the Test for Mathematical Concepts for the Seventh, Eighth, and Ninth Grades.

The Distributions of the Scores for Grades Seven, Eight, and Nine

The scores for grade seven range from 1 to 46; those for grade eight range from 11 to 53; and those for grade nine range from 5 to 58. The mode in each case is from two to five score-points higher than the median is, while the mean is in each case slightly lower than

the median. The degree of concentration of the measures about the measures of central tendency can probably best be shown by the standard deviations of the distributions. These range from 8.16 units to 10.03 units.

The distributions are not entirely symmetrical but are negatively skewed. This persistent tendency of the scores to pile up toward the upper ends of the distributions is observed both in the general grade distributions and in the grade-sex distributions. Computations of skewness yielded negative results for all nine distributions. These results are shown in Tables VII, VIII, and IX.

The Overlapping of the Distributions

The amount of overlapping of the distributions for the three grades is large. The highest score in the seventh grade distribution is only 9 points below the highest score in the ninth grade distribution, and is 41 points above the lowest score in the ninth grade distribution. The lowest ninth grade score is only two points above the lowest seventh grade score. The eighth grade scores range from 10 points above the lowest seventh grade score to 5 points below the highest ninth grade score.

The eighth grade median was reached or surpassed by 26 per cent of the seventh grade pupils, and the ninth grade median was reached or surpassed by 15 per cent of the eighth grade pupils. On the other hand, 24 per cent of the ninth grade pupils scored below the eighth grade median and 23 per cent of the eighth grade pupils scored below the seventh grade median. The ninth grade median was reached or surpassed by 4 per cent of the seventh grade pupils, whereas the scores of 11 per cent of the ninth grade pupils did not exceed the seventh grade median.

The inter-quartile ranges for grades seven and nine do not overlap but are separated by only .27 of a score-point. The inter-quartile range for grade eight overlaps the inter-quartile ranges of grade seven and grade nine by 5.38 score-points and 4.38 score-points respectively.

A range of $\pm 3.0\ \sigma$ from the mean of any grade distribution will include practically the entire range of scores for any other grade distribution. The range of scores within any given grade is several times as great as the range between the measures of central tendency of any two grades, and the standard deviation for any grade is greater than the average gain in any grade-to-grade interval.

Chapter III

MASTERY OF THE INDIVIDUAL CONCEPTS AND OF CERTAIN SUB-GROUPS OF CONCEPTS

The Second Problem

In the first part of this investigation the attempt was made to ascertain the extent to which the group of sixty-three concepts as a whole had been mastered by pupils in various groups and at different grade levels of the junior high school period. In this second part the study will be directed toward the problem of finding out at what levels and to what extent the individual concepts and certain sub-groups of concepts have been mastered by these pupils.

The Technique Employed

In order to make this approach to the analysis of the data it was necessary to construct an entirely new series of charts. In each of these charts the 63 concepts were listed in order on the horizontal scale, while the case numbers were listed consecutively on the vertical scale.[20] Thus there was one cell in the chart for each concept in the test for each pupil who took the test. If a particular pupil made the correct response to a particular concept in the test, the fact was indicated by placing a check mark (√) in the appropriate cell in the chart. If that pupil did not give the correct response to that item, that particular cell in the chart was left blank. One such chart was made for each grade-group in each school. These charts provided systematic checking space for 104,454 items.

All of the correct responses on the 1,658 tests were thus systematically tabulated from the test papers, and each tabulation was subsequently completely checked. The total number of correct responses on all of the charts was found by actual count to be 51,950. To insure accuracy this number was checked against the total number of correct responses, as taken from the other series of tabulation sheets used in studying the data for the first section of this report. The cross-count checked exactly with the count from the original tabulations, not only as a whole but in each one of the sixty-four primary grade-sex groups.

From this new series of charts it was possible to determine the

[20] Illustrative samples of forms used were included in the Appendix of the original study on file in the library of the University of Missouri.

exact number of correct responses to each item in the test for each one of the grade-sex groups. These totals were re-compiled into whole-grade totals for each school, then into grade-sex group totals and whole-grade totals for each of the three classifications of schools, and finally into grade-sex group totals and whole-grade totals for all the schools together.

Since the number of pupils in each of these sub-groups was known it was possible to compute for each test item and for any group or sub-group of pupils the per cent of the pupils giving a correct response. Thus it became possible to locate any particular group of pupils on a "per cent of mastery" scale with reference to any given concept in the list.

The per cent of pupils in a given group who responded correctly to a given item will be called the "coefficient of mastery" or the "mastery coefficient" for that group on that item. For example, 33 per cent of the seventh grade pupils in the S-Group responded correctly to item 1. The "coefficient of mastery" for item 1 for grade seven in the S-Group is therefore said to be 33. Coefficients of mastery were computed for each concept for each grade in each of the three separate classifications of schools and for each grade when all the schools were thrown together into one general composite group. A total of seven hundred and fifty-six such mastery coefficients are thus available for study.

Mastery of Individual Concepts by the Three Classifications of Schools

The three classifications of schools exhibit in each grade a marked consistency in scoring for the individual items in the test. Since the numbers of correct responses on the various concepts have all been translated into per cents, or coefficients of mastery, it is possible to make direct comparisons of the performance of the small, medium, and large schools for each grade and for each item in the test.

It will facilitate discussion to designate the three per cents or coefficients representing respectively the locations of a given grade for the large, medium, and small schools on a "per cent of mastery" scale as "a set" of mastery coefficients. Since there are three grades under consideration there are, correspondingly, three of these "sets" for each concept, or 189 such "sets" in the whole table. For the moment attention will be directed toward the comparison of the individual coefficients within the various sets.

A careful study of the facts showed that in 158 (83.6 per cent) of the 189 sets the range between the best performance and the poorest performance does not exceed 25 units on a 100-unit scale of possible performance. In other words, in more than four-fifths of all the sets the difference between the best and the poorest actual performances does not exceed one-fourth of the possible difference. Moreover, in 73 per cent or almost three-fourths of the sets, the range is not more than one-fifth of the possible difference. In 38.1 per cent or approximately two-fifths of the sets the range does not exceed 10 units (one-tenth of the possible difference), and in 16.4 per cent or one-sixth of the sets the coefficients are all within 5 units of each other, or are concentrated within one-twentieth of the whole scale. In only one-sixth (16.4 per cent) of all the 189 sets does the range of performance cover more than one-quarter of the scale, and in only two cases (one-half of 1 per cent) is the difference between the best and the poorest performances greater than half of the scale.

It seems worthy of note that such a degree of compactness of scoring should be found within so many of the sets, the more so in view of the fact that the sets themselves range all over the scale of possible mastery. It appears that schools of different sizes tend to achieve very similar results with respect to mastery of most of the individual concepts, just as they do with the list as a whole, as was pointed out in the first section of this report.

Distribution of Concepts According to Per Cent of Mastery, All Schools Considered Together

Table X shows the number of test items for each grade in each tenth of the "per cent of mastery" scale.

The data of Table X make it apparent that there is a substantial general gain in the mastery of individual concepts from grade to grade. In grade seven only 23 of the concepts, or a little over one-third of all of them, fall in the upper half of the mastery scale. This proportion is increased to 32 concepts, or slightly over half of the group, for grade eight, and is further increased to 44 concepts, or more than two-thirds of the whole group, for grade nine.

Ranking of the Concepts in the Three Grades

A slightly different approach to the problem of improvement in mastery of the individual concepts may be made by studying the ranking of the coefficients of mastery in the three grades. The ranks of the

TABLE X
Distribution of the Concepts on the "Per Cent of Mastery" Scale

Per Cent of Mastery	Number of Concepts		
	Grade 7	Grade 8	Grade 9
0–10	8	5	0
11–20	7	5	5
21–30	10	6	3
31–40	8	9	5
41–50	7	6	6
51–60	4	8	3
61–70	10	5	12
71–80	7	10	16
81–90	1	6	7
91–100	1	3	6
Total	63	63	63

Table reads: The number of concepts mastered by 10% or less of the pupils in grade seven is 8; in grade eight, 5; and in grade nine, 0. The other rows in the table are to be read in a similar manner.

coefficients of the 63 individual concepts for each of the three grades were worked out and the concepts were listed in accordance with the ranking of their coefficients for grade nine, but the three coefficients for each item were placed side by side so that the range between the high and the low ranks for a given concept could readily be determined.

A study of the data reveals a noticeable consistency in the ranking of individual concepts in the three grades. In general those concepts which rank high for grade nine tend to rank high for grades seven and eight also, and those which rank low for grade nine tend to rank low for the other two grades, although individual exceptions to this will be found.

The Rank-Ranges

It is observed that in only one case is the "rank-range" (or the disparity between the two extreme rankings for a given concept) in excess of one-half of the total of 63 places. Only two of the 63 concepts have rank-ranges of more than 25 places, while 16 concepts have rank-ranges of more than 10 places. On the other hand 47 concepts have rank-ranges of 10 or less places; 18 of these have rank-ranges of 5 or less, and in 10 cases the range does not exceed two places. For four concepts the range is only one point, and one concept has exactly the same rank in all three grades. The median rank-range is only 8.07 $\pm$ .42 places.

TABLE XI

Distribution of Total Rank Ranges

Rank-Range	No. of Concepts
5 or less	18
6–10	29
11–15	8
16–20	2
21–25	4
26–30	1
31–35	1
36 or more	0
Total	63

Table reads: The number of concepts having rank ranges of 5 or less is 18, etc

Comparison of the rankings in the seventh and eighth grade groups reveals that the greatest displacement of any concept is 14 places and the second greatest displacement is only 10 places. Only 14 concepts have displacements greater than 5 places, and only two have displacements as great as 10 places. For 49 of the concepts the displacement is 5 places or less, and the median displacement is only 3.31 $\pm$ .36 places. Table XII gives a summary of the displacements among the seventh and eighth grade rankings of the concepts.

TABLE XII

Displacements Among Seventh and Eighth Grade Rankings

Displacement	No. of Concepts
10 or more places	2
More than 5 places	14
Exactly 5 places	5
Exactly 4 places	7
Exactly 3 places	8
Exactly 2 places	13
Exactly 1 place	10
No displacement	6
Total	63
Median Displacement	3.31 $\pm$ .36 places

The correspondence of the rankings in the eighth and ninth grade groups is not quite as close as that found when the seventh and eighth grade groups were compared. Thirteen concepts have displacements of 10 or more places and almost half of the concepts have displacements of more than 5 places. For 36 of the concepts the displacement is 5

places or less. The median displacement is $4.83 \pm .55$ places. Table XIII gives a summary of the displacements among the eighth and ninth grade rankings of the concepts.

TABLE XIII

Displacements Among Eighth and Ninth Grade Rankings

Displacement	No. of Concepts
10 or more places	13
More than 5 places	27
Exactly 5 places	4
Exactly 4 places	3
Exactly 3 places	2
Exactly 2 places	13
Exactly 1 place	11
No displacement	3
Total	63
Median Displacement	$4.83 \pm .55$ places

Two general conditions have now been shown to exist. In Chapter II it was shown that there is a general improvement from grade to grade in mastery of the group of concepts as a whole. In the present chapter the data indicate that there is a general grade-to-grade improvement in mastery of individual concepts and that the ranks of the mastery coefficients of the individual concepts tend to vary but slightly from grade to grade.

These conditions may be broadly interpreted together to indicate that under prevailing conditions of instruction and learning, the process of mastery of the concepts is a gradual one, extending with some considerable uniformity over the whole junior high school period and over practically the entire list of concepts considered in this study. The uniformity of improvement seems to be more pronounced below grade nine than during grade nine. Apparently there is some shift in emphasis after the end of the eighth grade. This would be expected. It should be born in mind, of course, that what has been said here is only a general statement of a general situation, and that many individual exceptions to uniform progress will be found.

Classification of the Concepts

For purposes of further study the concepts were arbitrarily classified by the investigator according to certain general divisions of the field of elementary mathematics to which they seemed most par-

ticularly to belong.[21] The extent to which these various sub-groups of concepts had been mastered was investigated. The divisions used in the classification, with the number of concepts in each, are shown in Table XIV.

TABLE XIV
Classification of the Concepts

Classification	No. of Concepts
Geometric concepts	25
Algebraic concepts	13
Business concepts	7
Arithmetic concepts	2
Graphic concepts	3
Trigonometric concepts	3
Measurement concepts	4
Function concepts	3
Concepts of equality	2
Unclassified concepts	1
Total	63

Group Coefficients of Mastery

To facilitate comparison it was necessary to establish a single mastery coefficient (per cent) for each grade in each classification of concepts. This was defined simply as the arithmetic mean of the mastery coefficients of all the individual concepts in the given classification and the given grade. These averages are used without qualification as rough indicators of levels of mastery for the groups.

A single general average mastery coefficient was computed for each of the ten sub-groups of concepts by weighting the three coefficients for the separate grades roughly in accordance with the numbers of pupils in the respective grades and taking the mean of these three weighted grade-coefficients. By using these weighted average coefficients, it was possible to rank the ten classifications of concepts with respect to the general extent to which they had been mastered by all the pupils taking the test.

Table XV presents a summary of the group mastery coefficients for the three grades, the weighted average coefficients, the ranks, and the grade-to-grade gains for the ten sub-groups or classifications of concepts.

[21] A classification of mathematical concepts along somewhat similar lines is found in Smith and Reeve, *The Teaching of Junior High School Mathematics*, pp. 43-47.

TABLE XV

Group Mastery Coefficients, Ranks, and Grade-to-Grade Gains for the Ten Classifications of Concepts

Classification	Mastery Coefficients			Grade-to-Grade Gains		Weighted Average	Rank
	Grade 7	Grade 8	Grade 9	7–8	8–9		
Business	55.1	63.8	69.6	8.7	5.8	62.4	1
Geometric	55.4	63.4	67.1	8.0	3.7	61.6	2
Graphic	47.3	58.0	66.0	10.7	8.0	56.5	3
Arithmetic	40.5	56.0	63.5	15.5	7.5	52.6	4
Unclassified	33.0	49.0	71.0	16.0	22.0	49.6	5
Equality	27.5	41.0	76.5	13.5	35.5	46.3	6
Measurement	33.8	43.0	51.0	9.2	8.0	42.0	7
Algebraic	21.1	32.4	61.2	11.3	28.8	36.6	8
Function	28.0	32.0	43.3	4.0	11.3	33.8	9
Trigonometric	10.7	8.3	16.3	−2.4	8.0	11.4	10

Table reads: For the Business Concepts the mastery coefficients are 55.1 for grade seven, 63.8 for grade eight, and 69.6 for grade nine, grade eight gaining 8.7 points over grade seven and grade nine gaining 5.8 points over grade eight on the "per cent of mastery" scale. The weighted average coefficient of mastery for the Business Concepts is 62.4 and this group ranks highest among the ten classifications The data for the other classifications of concepts should be read in a similar manner.

The Business Concepts

There are seven concepts in this list. They are: *loss, simple interest, taxation, insurance, profit, compound interest,* and *net price.* These are all concepts which are directly related to money. Most of them find frequent application in the ordinary experiences of most people, and are commonly involved in ordinary discussion of business matters. Moreover, they are common to most textbooks in seventh and eighth grade mathematics. It is therefore not surprising that they should be familiar to a considerable proportion of pupils at the seventh grade level or above. A consistent gain is found in each grade-to-grade interval.

The Geometric Concepts

This group ranks second among the ten classifications. It includes 25 concepts and is thus more than twice as large as any other classification except that of Algebraic Concepts. The concepts which are included are the following: *similarity, volume, circle, rectangular prism, surface, trapezoid, pyramid, cone, sphere, perpendicular lines, congruence, cylinder, triangle, parallelogram, square, cube, parallel lines, length, rectangle, a drawing to scale, perimeter, area, angle, vertex,* and *pi.*

Sixteen of the twenty-five concepts listed above are concepts of geometric objects or figures or arrangements of lines. Most of these are of such common occurrence that practically every person sees them daily. Moreover, the names of many of them (e.g., circle, square, cylinder, triangle, etc.) are often used in conversation without any thought of their being misunderstood. It seems natural that such concepts should have a rather righ coefficient of mastery. The mastery coefficients of this group of sixteen concepts of geometric form are given in Table XVI.

TABLE XVI

Mastery Coefficients for Concepts of Geometric Form

Grade 7	57.9
Grade 8	66.5
Grade 9	71.4
Weighted Average	64.8

It will be advantageous to carry the analysis of this group of geometric concepts one step further by comparing the mastery of the concepts of plane geometric forms with that of the concepts of solid geometric forms. The coefficients of mastery are shown in Table XVII.

TABLE XVII

Mastery Coefficients of Plane and Solid Geometric Forms

	Plane	Solid
Grade 7	55.5	61.8
Grade 8	63.2	72.0
Grade 9	67.8	77.5
Weighted Average	61.8	69.9

It is immediately seen that the concepts of three-dimensional figures show a decidedly higher degree of mastery in all three grades than do the concepts of two-dimensional figures. It might be supposed that two-dimensional figures would be simpler and more easily comprehended than three-dimensional figures, but the data do not support such an hypothesis. It is possible, of course, that this hypothesis may be perfectly true and yet be overbalanced by some other factor more potent than mere relative simplicity. In any event, it appears that on the whole, children do master the concepts of three-dimen-

sional forms to a greater extent than they do those of two-dimensional forms.

The remaining nine geometric concepts in the list have been designated "Concepts of Geometric Properties." By this term is meant geometric concepts which are not, themselves, geometric figures but which are fundamentally associated with geometric figures and cannot well be though of independently of such figures. The list includes the concepts of *similarity, volume, surface, congruence, length, perimeter, area, vertex,* and *pi.*

A word of explanation of the reason for including *pi* in this list may be in place. *Pi* may be regarded as being intrinsically a ratio and therefore a number. On the other hand it has no significance and no use in junior high school mathematics except in determining certain characteristics and properties of a circle. Its meaning may thus be regarded as essentially geometric, and for this reason it has been included in the list of geometric properties.

The composite mastery coefficients of this group of concepts of geometric properties are shown in Table XVIII. These mastery coefficients are noticeably lower than those of either of the groups of geometric forms.

TABLE XVIII
Mastery Coefficients of Concepts of Geometric Properties

Grade 7	51.0
Grade 8	57.9
Grade 9	59.5
Weighted Average	55.8

The concept of *congruence* ranks lowest in the list with coefficients of 7, 12, and 11, respectively, for the three grades. The next higher one is *similarity* with coefficients from two to three times as large as those for *congruence.* These two concepts, and particularly that of *congruence,* ordinarily are not formally encountered until the tenth grade is reached. Neither term, in its technical sense, is current in common non-mathematical usage.

It appears that there is little effective teaching of these concepts of *geometric properties* during the ninth grade since the general ninth grade coefficient is less than two points higher than that for the eighth grade.

This study of the mastery of geometric concepts may be sum-

marized by saying that the concepts of three-dimensional forms rank highest, those of two-dimensional forms next, and the concepts of geometric properties rank lowest. The geometric concepts as a whole rank second only to the concepts of business arithmetic and the difference between these two groups is very small.

Concepts Related to Graphs

There are only three concepts in this list: *a graph, a statistical graph,* and *graphic representation.* The apparent similarity of the three concepts makes it difficult to understand why there should be any material disparity between them so far as mastery is concerned, yet a marked disparity is found. The coefficients for the concept of *a statistical graph* were only about half as high in all grades as those for the concept of *a graph. Graphic representation* also ranged far below *a graph* except in grade nine.

At all events, these concepts related to graphs stand third among the ten classifications, and in point of general average, score only a few points below the Business Concepts and the Geometric Concepts. As a matter of fact, graphs are somewhat closely related to geometry, and statistical graphs, at least, are frequently used in connection with discussions and writings concerning business and financial matters. It is perhaps not at all strange that this group of concepts should rank close to the Business Concepts and the Geometric Concepts.

The Arithmetic Concepts

This group really includes only two of the nine concepts which might have been included under this caption. The other seven, however, seemed to form a closely centered sub-group and were studied separately as Business Concepts. Under the heading "Arithmetic Concepts" only the concepts of *ratio* and *average* are included. It will be observed that each of these represents either an expressed or implied relation of numerical quantities.

These two concepts score fairly high in all the grades, and show consistent, though not entirely regular, increases from grade to grade. As a group, however, they rank fourth among the ten classifications, although it should be noted that the difference between the weighted coefficient of this group and that of the highest ranking group is slightly less than one-tenth of the entire mastery scale.

Concepts of Equality

The concepts of equality rank sixth among the ten groups. This group includes only two concepts: namely, the concept of *equality* and the concept of *an equation*. The gain of this group during the eight-nine grade interval exceeds the grade-to-grade gain of any other group in any single interval, and the degree of mastery attained in the ninth grade exceeds that for any other group in any grade. The mastery curve for the concepts of equality parallels rather closely that for the algebraic concepts but is from six to fifteen points higher. Like the algebraic concepts, this group of concepts of equality exhibits a gain in mastery during the ninth grade which is more than twice as great as the gain during the eighth grade. The foregoing data suggest the probability that concepts of equality receive considerably more emphasis or use in beginning algebra than they do in arithmetic as these subjects are now taught.

Concepts of Measurement

This group includes the four concepts of *indirect measurement, a measurement, an error in a measurement,* and *an approximation in a measurement*. As a group these concepts rank seventh among the ten groups. The concept of *indirect measurement* has mastery coefficients in all three grades almost twice as high as those for the concept of *a measurement*. The concept of *an error in a measurement* has lower coefficients than any of the other three concepts in this group. This tends to confirm the empirical observation of the author that as a rule pupils are not effectively taught to regard every measurement as involving some error, though the error may be extremely small.

In view of the importance which now is generally attached to measurement as an element of junior high school mathematics, it would seem that a general mastery coefficient of less than 50 per cent on these concepts related to measurement may reasonably give concern to all who are interested in the teaching of junior high school mathematics.

The Algebraic Concepts

By the term *Algebraic Concepts* is meant those concepts in the list which ordinarily are first encountered in their general form in the study of algebra or which are used mainly in connection with situations that are essentially algebraic, rather than arithmetic or geo-

metric, in nature. This group includes 13 concepts. They are as follows: *root of an equation, formula, coefficient, a solution of an equation, exponent, positive and negative numbers, algebraic fraction, algebraic numbers, algebraic product, algebraic factor, a quadratic, a root of a number* and *power of a number.*

The coefficients indicate clearly that there is a very substantial gain in mastery where it would reasonably be expected: that is, in the ninth grade. Most of these concepts are technical and are peculiar to algebra. Since algebra is usually taught formally for the first time in grade nine, it is quite natural to expect the ninth grade mastery coefficient to be materially larger than those for the seventh and eighth grades. This is found to be the case.

The concept of a *formula* is the only one of the thirteen which was generally mastered by all three grades. This is probably because formulas are frequently used in arithmetic. While in its general sense "formula" is undoubtedly an algebraic concept, as a matter of fact, specific formulas are so frequently used in arithmetic (especially in mensuration) that it might be about as easy to defend the classification of this concept with the Arithmetic Concepts as with the Algebraic Concepts.

The mastery of the other concepts is of course not uniform in any given grade. On the contrary, very considerable diversity is the rule. The individual concepts all show consistent increases from grade to grade, however, and in two-thirds of the cases the improvement of grade nine over grade eight is more than twice as much as the improvement of grade eight over grade seven. This group ranks eighth among the ten groups.

Concepts of Functionality

The concepts in this group may be regarded as being rather closely allied to the Algebraic Concepts. In the opinion of the author, however, there is a distinction to be made, because all of the concepts in this group are used frequently in other than algebraic situations. The characteristic which they seem to have most in common is that they all refer to the functional relation between quantities. Hence this group has been called Concepts of Functionality.

The concepts which are included are those of *dependence, direct variation,* and *proportion.* As a group these concepts rank ninth among the ten classifications. It is worth noting that the concept of *direct variation* ranks far below the other two, even in the ninth grade. As

a matter of fact, its ninth grade coefficient is lower than its eighth grade coefficient. In view of the recommendation contained in the report of the National Committee on Mathematical Requirements[22] the low ranking of this group of concepts may be regarded as a serious indictment of existing instruction in elementary mathematics.

The Trigonometric Concepts

The only three essentially trigonometric concepts rank as a group far below the other groups. The list includes only the concepts of the *sine, cosine,* and *tangent* of an angle. Children would ordinarily have practically no opportunity of learning these concepts incidentally out of school, and it appears that they find little place in the formal instruction in junior high school mathematics. The only other plausible explanations of the low degree of mastery are that the teaching of these concepts is very poorly done, or else that the concepts themselves are of such difficulty that adequate mastery of them requires intellectual maturity above the level of that possessed by junior high school pupils. It is of interest, though perhaps not of much significance, to note that grade seven shows a slightly higher mastery of each one of these three concepts than does grade eight.

The Unclassified Concept of Algebra as a Tool of Science

It seemed impossible to classify this concept appropriately in any of the groups that have been listed. It is not particularly an algebraic concept, but rather, it is the concept of one of the functions or purposes of algebra. Therefore it has been listed separately. It is interesting to observe that a general and somewhat abstract concept such as this can be mastered to a relatively high degree by pupils of junior high school age. It exhibits a general improvement in mastery from grade to grade, and in each grade-to-grade interval this improvement appears large enough to be meaningful.

Summary of the Chapter

It is evident that in most of the ten classifications all sorts of variations are found in the mastery of individual concepts by the various grades. Some concepts have high coefficients throughout, some have low coefficients throughout. For some concepts grade eight shows large

[22] *Op cit.*, Chapter VII.

increases over grade seven with little improvement being made in grade nine. For others the improvement of grade nine over grade eight is much larger than the improvement of grade eight over grade seven. For some concepts very substantial gains are made in both grade-to-grade intervals. A few show actual grade-to-grade regressions.

Any general statements made about the groups must be considered only as broad averages. Aside from the single unclassified concept, the group of trigonometric concepts is the only one in which the mastery coefficients of all the individual concepts closely approximate the group coefficients.

Comparisons reveal that the Business Concepts have higher mastery coefficients than any of the other classifications. The Geometric Concepts rank only very slightly below the Business Concepts in all grades, and the weighted mean coefficient for the Geometric Concepts is less than one point below that for the Business Concepts. Next, and not far below the Geometric and the Business Concepts stand the Concepts Related to Graphs, and the Arithmetic Concepts. The weighted coefficients for these two groups are respectively about six points and ten points below that for the Business Concepts.

The single unclassified concept ranks fifth among the groups, with the Concepts of Equality and the Concepts of Measurement holding sixth and seventh places respectively. The Algebraic Concepts rank eighth, the Concepts of Functionality ninth, and the Trigonometric Concepts last. The weighted mean coefficient for the Trigonometric Concepts is more than 22 points below that for the next higher group, and is 51 points below the weighted mean coefficient for the Business Concepts which head the list. As might be expected, the mastery coefficients of the Trigonometric Concepts are found to be relatively very low in all three grades. Moreover, this is the only one of the ten groups which exhibits a grade-to-grade regression.

Chapter IV

THE RELATION OF MENTAL AGE AND OF CHRONOLOGICAL AGE TO MASTERY OF THE MATHEMATICAL CONCEPTS

It is obvious that there are many things which might be influential in affecting the extent of mastery of mathematical concepts. The sum-total of the environmental experiences of the pupil is doubtless an important factor, but unfortunately this is such a complex thing that it cannot be completely described, much less evaluated. The present part of this study will be directed toward the problem of finding out what relation exists between mastery of the concepts and two potential conditioning factors: chronological age and mental age. The influence of environmental factors will affect the results only to the extent to which they may enter into the determination of mental age and of performance on the test of mathematical concepts. The technique which will be used in this part of the study is that of partial correlation.

Through the courtesy of one of the co-operating schools, intelligence quotients were furnished for 279 pupils. These intelligence quotients, together with the chronological ages and the scores of the pupils on the test of mathematical concepts, constitute the basic data for this section of the report.

A Necessary Assumption

It seems obvious that any inference drawn from a study of the relation between test scores and intelligence quotients *per se* would be vitiated by inequalities of chronological age. What was purposed was to find out the relation between mastery of the concepts and absolute mental maturity, as measured by mental age, rather than relative mental maturity. The primary data contained no direct indication of the mental ages of the pupils at the times when the intelligence tests were given, but merely indicated the intelligence quotients. It therefore became necessary to translate these into mental ages.

The process by which this was accomplished involves one fundamental assumption, which is that the intelligence quotient remains constant for a given individual. The results of this part of the study must be viewed in the light of this assumption. That it probably is not strictly true is at once admitted, but there is abundant evidence

that the intelligence quotient of a given individual tends to remain at about the same level.

On the basis of this fundamental assumption it was possible to compute the mental age of each of the 279 pupils. This was done by a simple algebraic transformation of the formula used in the computation of the intelligence quotient.

$$\text{I.Q.} = \frac{\text{M.A.}}{\text{C.A.}}\text{; therefore M.A.} = \text{I.Q.} \times \text{C.A.}$$

Other Limitations

In addition to the assumption mentioned above, the investigator is aware of certain other limitations which should be acknowledged.

In the few cases where actual chronological age was higher than 192 months, it was taken as 192 both for calculating the mental ages and for determining the correlation between test scores and chronological age. All the intelligence quotients were based upon data secured from the *Terman Group Test of Mental Ability,* and the computation of intelligence quotients based upon this test admits of no measure of chronological age in excess of 192 months. In order to be consistent, therefore, it was necessary to follow this same rule in the computation of mental ages from intelligence quotients and chronological ages.

In the correlation of chronological ages with test scores it would have been possible to use the actual chronological ages in all cases, instead of using 192 as a maximum. On the other hand, had this maximum not been observed there would have been involved the other apparent inconsistency of using one set of chronological ages to compute mental ages and another set to correlate with mental ages and test scores. It was felt that this latter inconsistency would probably be harder to justify than the former one, and so 192 was used as the maximum chronological age throughout.

The Relation of Mental Age to Mastery of the Concepts

The group of individuals used in this part of the study included 279 pupils, both boys and girls, in grades seven, eight, and nine. The scores of this group on the *Test of Mathematical Concepts* ranged from 3 to 56, covering almost the whole gamut of possible scores. Wide spans were also found between the extremes of mental age and of chronological age, the mental ages ranging from 115 months to

269 months, and the chronological ages ranging from 135 months to 192 months.

The simple coefficient of correlation between mental age and test scores (by which is meant scores on the test of mathematical concepts) was found to be $.50 \pm .03$. When the influence of chronological age was "partialed out" by statistically holding chronological age constant, the coefficient of partial correlation between these two variables was found to be $.47 \pm .03$.

There appears to be a definite direct relation between mental age and the mastery of mathematical concepts, and this relation appears to be only slightly affected by the factor of chronological age, since its statistical elimination affected the coefficient only slightly.

The Relation of Chronological Age to Mastery of the Concepts

A different situation was found with respect to the relation of chronological age to test scores. The investigator had rather expected to find a direct relation between these two variables but the data give no support to such an hypothesis. As a matter of fact, the indications are that a negative relation exists between these two variables.

The simple coefficient of correlation between chronological age and test scores was found to be $-.28 \pm .04$. In order to free this coefficient from the influence of variability of mental age, the technique of partial correlation was again employed. When the factor of mental age was thus statistically held constant the coefficient was reduced to $-.22 \pm .04$.

Care must be taken to avoid an erroneous interpretation of this coefficient. As the author sees it, one can hardly infer that mere growing older causes a given pupil to lose his mastery of some of the concepts. Rather, one may infer that of a large group of pupils all of whom have attained a given mental age, those who have attained this mental age early in life will be likely to have attained mastery of more of the mathematical concepts than those who have attained the given mental age after a longer chronological period.

Conclusion

It seems that in general the pupil with a low I.Q. not only masters mathematical concepts with less chronological rapidity than the pupil with a higher I.Q., but at the same mental age has actually mastered fewer of them. In other words, it would appear that the pupil of low I.Q. tends to increase his mastery of the mathematical con-

cepts less rapidly than he increases in mental age as a whole, while the pupil of high I.Q. tends to increase his mastery of the mathematical concepts more rapidly than he increases his mental age as a whole.

If this were the case we should find a positive but not a perfect correlation between intelligence quotients and test scores. This is exactly what does exist, for the simple correlation between I.Q.'s and test scores is precisely what was determined when the effect of chronological age was partialed out and the partial correlation of mental age with test scores computed, because with chronological age held constant the formula for computation of the intelligence quotient becomes I.Q. $= k \cdot$ M.A. and $r_{12} = r_{1\ (2k)}$.

On the basis of the foregoing considerations it is reasonable to conclude that the degree of mastery of these mathematical concepts is directly associated with mental age. The data suggest that attained chronological age, in and of itself, bears an inverse relation to mastery of mathematical concepts, but the absolute value of the coefficient indicating this relation is only about half as great as that of the coefficient indicating the degree of association of test scores and mental age. One may fairly conclude that there is reason to expect pupils of high intelligence to master mathematical concepts rapidly, and to expect pupils of low intelligence to master mathematical concepts slowly. In either case the data give no reason to expect the mere factor of increasing chronological age to effect any improvement in mastery of mathematical concepts.

The coefficients of partial correlation furnish the means for setting up a regression equation which may be used in predicting probable scores on the test of mathematical concepts. Letting x_1 = probable test score, x_2 = M.A. in months, and x_3 = C.A. in months the equation becomes:

$$x_1 = .223\, x_2 - .138\, x_3 + 10.97$$

or

$$x_1 = \frac{x_2 - .619\, x_3 + 49.189}{4.484}$$

By means of the above equation it is possible to predict, within certain limits, the most probable score of a pupil on the test of mathematical concepts when his mental and chronological ages are known.

Chapter V

SUMMARY AND SUGGESTIONS

This final chapter is intended to serve three purposes:

(1) To bring together in concise form those findings of the investigation which seem to the author to be significant;

(2) To suggest possible interpretations and conclusions based upon certain of these findings; and

(3) To suggest possible lines of investigation along which extensions of the present study might profitably be made.

These three parts of the chapter will be presented in the order in which they have been listed.

Summary of the Findings of the Investigation

1. In every grade-to-grade interval in each grade-sex group in each of the three classifications of schools the group averages show a definite gain in mastery of the concepts. There is not a single exception to this.

2. Grade medians reveal that the average pupil completing the seventh grade has mastered a little over one-third of the 63 concepts included in this investigation and that the average pupil completing the ninth grade has mastered about two-thirds of these concepts. In other words the average pupil probably enters the junior high school period with a mastery of not more than roughly one-third of the concepts, and leaves it with about one-third of them still unmastered.

3. The medians indicate that grade-to-grade improvement in mastery is substantially uniform so far as grades seven, eight, and nine are concerned. While there are irregularities in the curve of medians, it is sufficiently regular to justify the inference that something approximating a direct linear relationship exists between grade level and degree of mastery of the concepts, at least through grade nine. If the mid-grade medians are omitted the approximation to linearity is more pronounced.

4. The general medians are fairly representative of those for each of the three classifications of schools. The widest departure is in the case of the group of medium-sized schools which stands out as distinctly superior in the seventh and the ninth grades, although not in the eighth. The range of medians in each grade for different sized schools is comparatively small.

5. The data of this investigation give no evidence of general superiority of either the junior high school type of organization of the school or the traditional 8-4 type of organization as regards mastery of the mathematical concepts. The school with the best record has a junior-senior high school organization, while the school with the poorest record is a junior high school. The medians for the only remaining junior high school in the group are very close to the general medians for the whole group of schools.

Comparatively little difference is found between the corresponding grade-medians for the entire group of junior high schools and those for the entire group of schools having the 8-4 grade organization. In both the seventh and the eighth grades the differences between the medians are less than half of one score-point, while in the ninth grade the medians differ by 1.47 score-points. The critical ratios show that in no case is there a certainty of a true difference between the corresponding grade medians.

6. According to the data presented, mastery of the concepts seems to proceed with considerable regularity up to about the end of grade eleven. There is some reason to believe, however, that the medians for the senior high school groups are too high to represent properly a normal situation in these grades.

7. In general the medians indicate that the degree of mastery of the concepts attained by boys is slightly greater than that attained by girls, and boys tend to increase their mastery of the concepts slightly more rapidly than girls do. The differences in average mastery are not large, however, and are not statistically certain in any one of the three grades, although the probability of a true difference is somewhat greater in the eighth grade than in the seventh, and is greater in the ninth grade than in the eighth.

8. The scores of girls, as shown by the grade medians for the three classifications of schools, tend to vary somewhat more widely from their general grade medians than do the scores of boys.

9. The wide spread of scores in the various distributions is noteworthy. The range of scores in every grade covers almost the whole possible scale. When the whole group of schools is considered together, the smallest standard deviation found among the three grade distributions is greater than the largest difference between any two consecutive grade medians or means. It follows that there is a great amount of overlapping of the distributions. It is evident that mastery

of the concepts within any given grade is characterized by wide variability rather than uniformity.

10. The coefficients of mastery of the different concepts exhibit extreme diversity, some indicating almost complete mastery in all three grades, and others indicating almost complete lack of mastery in all three grades. Most of the individual concepts show progressive mastery in each of the grade-to-grade intervals, though this progress is generally irregular and there are a few concepts for which actual regressions are found. There is no concept for which a regression occurs in more than one grade-to-grade interval.

11. A very marked consistency is found among the corresponding coefficients of mastery on individual concepts for the three classifications of schools. The range for a given *set* of coefficients tends to be relatively small.

12. There is evidence of a general progressive gain in mastery of the individual concepts. This tendency embraces in general the entire list of concepts and the entire period covered in the main part of this study. It is manifested in the persistent grade-to-grade increases in the coefficients of mastery for the different individual concepts, and in the consistency in the ranking of the coefficients for the three grades. The combined evidence of these two phenomena leaves little room for doubt of the general proposition stated above.

13. Concepts which are normally encountered frequently in extra-school situations tend to be mastered more adequately than those which are not frequently so encountered. Thus, the concepts of business arithmetic (concepts related to money) and the concepts of common geometric forms head the list, while the trigonometric concepts stand far below all other classifications.

Some Tentative Conclusions

On the basis of the data presented in this report it is reasonable to conclude that there is evidence of fairly consistent and uniform improvement in mastery of the sixty-three mathematical concepts during the secondary school period, and particularly during the junior high school period. If one may assume, however, that one of the aims of mathematical instruction in the junior high school grades should be to produce complete mastery of these sixty-three concepts, there is equally conclusive evidence that current practice falls far short of its goal in this respect.

Whatever may be the processes through which the concepts are mastered it is clear that the existing educational agencies are making inadequate provision for the operation of these processes. This is attested by the extreme individual differences noted among the raw scores and by the fact that the average pupil, upon completion of the ninth grade has attained mastery of less than two-thirds of the concepts.

The evidence points to the probability that there is no material difference between the results secured in organized junior high schools and in schools which do not have the junior high school organization, nor does the size of the school system appear to make any great difference in the extent to which the concepts are mastered by the pupils.

The larger schools might be expected to have more adequate financing, material equipment and facilities for instruction than the smaller schools, and the junior high schools have at least theoretical advantages over schools which do not embody this form of organization, in that their teachers are supposed to be specially trained and selected and their courses of study more particularly designed to realize the valid objectives that have been set up for this educational period. Therefore it might be supposed that the larger schools and the junior high schools would show to better advantage than the other schools on the test of mathematical concepts. While the fact that they do not should not be taken as an off-hand indictment of them, still it gives a basis for raising a question as to whether they are making the largest possible use of the advantages enumerated above.

Some of the individual concepts seem to be mastered much more thoroughly than others. There are at least three possible explanations of this:

(1) difference in intrinsic difficulty of the concepts;

(2) difference in the emphasis placed upon various concepts in the formal instruction in mathematics; and

(3) difference in the extent to which pupils have the opportunity and incentive to master the various concepts in their extra-school experiences.

It is not improbable that all three of these factors, and possibly others, are operative in varying degrees. The data of this investigation give no absolute assurance of this, but there are some indications that it is probably true. In particular, the high ranking of the Business Concepts and the Geometric Concepts tends to substantiate the

hypothesis that out-of-school experience may be an extremely potent factor in the mastery of mathematical concepts.

Hypotheses and Suggestions

In conclusion the author wishes to propose and briefly discuss two hypotheses and to suggest certain problems which might profitably be the subjects of future investigation. These hypotheses and suggestions are not to be thought of as part of the investigation which has been reported, although they have been suggested by it. So far as the author knows, none of them have been directly subjected to experimental verification, although a study by Hydle[23] has some implicit bearing on the first two to be proposed.

There is reason to believe that mastery of mathematical concepts is an element of considerable importance in mathematical thinking, or the type of mathematical ability which Everett designates under the caption of "associative skills." This belief is entirely reasonable and is supported by well-qualified authorities. In particular, attention is directed to the quotation from the Report of the National Committee on Mathematical Requirements which was cited on page 3 of this report.

The diversity and the incompleteness of mastery of mathematical concepts characterize a situation which has certain implications believed to have important bearings on method in mathematical instruction. The present purpose of the author will be to point out and discuss briefly two of these which seem to him to be of large significance.

Rational Procedure versus Mechanical Procedure

One of the most frequent criticisms, and probably one of the most valid criticisms, of the mathematical work of children is that it is so largely mechanical. No one would deny that there are some processes which should be mechanized to a high degree of perfection, but no amount of mere mechanization can, in and of itself, produce an understanding of a problem situation. Yet the failure to bring about rational understanding of problem situations is precisely the charge that is being brought against mathematical instruction. To quote Everett[24] again:

[23] Hydle, L. L., *Elements of Difficulty in the Interpretation of Concrete Problems in Arithmetic,* Madison, Wisconsin, University of Wisconsin, 1925.

[24] *Op cit.*, p. 9.

. . . Enough is known . . . to offer abundant reason for the belief that much improvement would result . . . from a more specific recognition and emphasis upon the . . . fundamental skills of understanding.
We seem reluctant to teach outright the meanings of processes. We seem to have a feeling that it is undesirable, or possibly unethical, to make perfectly plain just what it is all about. . . .

It is the opinion of the author that reluctance has little to do with it. It is rather a matter of not knowning how to produce these "associative skills," these "recognitions of significance" which constitute understanding.

It may be that one of the reasons why pupils find it difficult to understand problems is that they do not understand the concepts involved. This proposition can be supported by empirical observational evidence, as well as by logic and by authority. One instance will be offered by way of illustration.

Many junior high school pupils have trouble in understanding the process of solving such a problem in percentage as this:

If 17 per cent of a number equals 51, what is 65 per cent of the number? There is reason to believe that the main cause, or at least one of the main causes, for failure to understand such a problem is the imperfect mastery of the concept of a per cent. When the same problem is stated in terms of familiar concepts the ratio of successes to failures invariably rises materially, and furthermore, the pupils are able to explain their understanding of their procedure.

For example:

If 17 ball gloves cost $51, how much will 65 of the same kind of ball gloves cost?

The pupil usually requires little help in understanding the process of solution. It appears quite reasonable to him that the first step is to find the cost of *one* ball glove and that this will be 1/65 of the total cost. This problem differs from the first one in no essential particular except that it is stated in terms of familiar concepts (ball gloves and dollars) instead of in terms of concepts of which the pupil has but an imperfect command. Yet a trial will convince the most skeptical critic that the difference in the results will be quite pronounced.

Verbal Problems

Verbal problems, or as pupils often call them, "thought problems," are at once the bugaboo and the backbone of a real course in elementary mathematics. Their extreme importance is attested by many

writers and critics of the teaching of mathematics, and the difficulty which they offer to pupils is emphasized with equal frequency.

It is the opinion of the present author that much of this difficulty would be obviated if the pupils had command of the fundamental concepts involved. A verbal problem is made up of concepts and relations between and among concepts. The task of the pupil is that of understanding the relationship (explicit or implicit) existing among the concepts so that he can set forth these relationships in precise mathematical form and thus arrive at his solution of the problem. But it is difficult to see how a pupil can intelligently set up, or even follow, mathematical relationships unless he has command of the fundamental mathematical concepts among which the relations exist and of the concepts of the types of relationships themselves.

In a word, it is believed that inadequate mastery of mathematical concepts may be responsible for a considerable amount of the difficulty which pupils encounter in their efforts to solve verbal problems. Therefore, it is believed that a good deal of this difficulty might be avoided if teachers would take more adequate measures to assure themselves that their pupils do have a mastery of such concepts before expecting them to use them. The author feels that those concerned with elementary mathematical instruction have been too liberal in their assumptions on this score, and that in the process of instruction the mastery of concepts may reasonably receive a share of deliberate (not merely incidental) attention comparable to that accorded to the mastery of skills.

Suggestions for Further Study

1. It would be desirable to have the problems studied in this investigation submitted to further experimental verification. It would be well to have a larger sampling of schools and to extend the basic parts of the study over a grade range which would include some grades below, as well as above, the junior high school period.

2. A study of the relative amounts of improvement in mastery of mathematical concepts in equal periods of time by pupils taking mathematics in school and by children otherwise comparable but not taking mathematics in school would be valuable.

3. A real contribution might be made by conducting an investigation of the relation of the mastery of mathematical concepts and the mastery of computational skills.

4. A similar study of the relation of the mastery of mathematical

concepts to success in the various phases of solving verbal problems would be extremely valuable.

5. A critical study of the relation of the mastery of the mathematical concepts to the mastery of the associative skills of mathematics would be especially significant.

6. It would probably be worth-while to investigate the relation of the mastery of mathematical concepts to certain major interests and aptitudes of pupils.

7. By carefully studying the relation of mastery of mathematical concepts to general success in certain schools subjects (e.g., physics, chemistry, advanced courses in mathematics, etc.) it is possible that it might be found to have considerable value as a prognostic instrument.

8. A study of the effects of different well-described methods of instruction in producing mastery of mathematical concepts and of the effects of varying the amount of emphasis on such concepts in the instruction would be of large practical value if it should develop that the mastery of such concepts is important.

9. There is need for a study of the proper grade-placement of mathematical concepts.

10. Eventually there will be a very definite need of a study which will review and integrate the findings of all specialized investigations relative to mathematical concepts, and which will offer broad interpretations and generalizations based upon sound scientific procedure.

There are doubtless many other questions related to the mastery of mathematical concepts, investigation of which would be of much value in enriching the present store of knowledge about their influence in effecting learning in various types of situations. It is the belief of the author that a series of studies extending the present investigation along the lines indicated above would ultimately make a very significant contribution to certain problems of curriculum and method in elementary mathematics.

René Descartes

Born in La Haye, France, March 31, 1596
Died in Stockholm, Sweden, February 11, 1650

"A man whose varied genius led philosophers to rank him primarily as one of themselves, physicists to claim him for their guild, and mathematicians to look upon him as one of the greatest geniuses in their domain, each group being fully justified in its opinion."—David Eugene Smith, *History of Mathematics,* I, p. 371.

A biographical note on René Descartes would naturally mention his friendship with Marin Mersenne* which dated from Descartes' school days in a Jesuit institution at La Flèche. It would speak of his years of travel, of his service in the army of Maurice of Nassau, of a short residence in Paris, of a period of study and writing in Holland (1628-1649), and of a brief career in Sweden at the court of Queen Christina. His philosophy, important as it is, does not concern us here.

Descartes lived at a time when there was tremendous interest in exact science. Physicists and mathematicians were occupied in quantitative investigations which had been impossible before the invention of such instruments as the telescope, the barometer, the thermometer and the airpump. Among the contributions of Descartes was the law of refraction of light. He propounded a theory of light and of color. He argued against the existence of a vacuum basing his opinion on the statement that the alleged vacuum at the top of a barometer could not exist for it was observed that light could pass through this space and, according to Descartes' hypothesis, light could be transmitted only through a material substance. Descartes' other investigations included the study of the mechanical advantage of simple machines—the pulley, wedge, lever, etc.

Galileo's trial before the Inquisition† took place while Descartes was in Holland. Nearly a century had passed since the publication (1543) of the great work of Copernicus on the heliocentric theory, but this study seems to have attracted little popular attention before the visible proofs of it were made possible through the invention of the telescope in the early part of the seventeenth century. The trial of Galileo made it evident that the earth was not to be summarily dis-

* See The Mathematics Teacher, October, 1931.
† The Mathematics Teacher, February, 1931.

placed from its position as an immovable center of the universe. The reaction of scholars varied. Mersenne, for instance, translated Galileo's work and so gave it greater publicity. Descartes propounded a theory of vortices characterized as being "intended to pay verbal homage to the geocentric formula while recognizing the heliocentric fact."*

In mathematics, Descartes was less diplomatic. He disputed various points with Roberval, Pascal, and Fermat and quite alienated these men. He not only said that anyone trained in mathematics could solve a particular problem which Roberval had just completed, but he submitted a proof which was far more concise than that of Roberval. He refused to believe that Blaise Pascal had written a particular essay on conics but declared that Pascal's father was pretending that his own work was the production of his son. It is not surprising that Descartes' influence was greater in Holland than it was in France.

Descartes' most important contribution to mathematics was his *Geometrie*,† published in 1637 as an appendix to his *Discours de la Methode*. In this work, the author shows how problems in geometry may be reduced to problems concerning the lengths of lines, designating some particular line segment as unity. He shows the correspondence between certain geometric figures and the sum, difference, products, quotients, etc. of quantities which are represented in general terms by letters. He excuses himself for the lack of detail by saying that by giving more, "I should deprive you of the pleasure of mastering it yourself, as well as of the advantage of training your mind by working over it, which is in my opinion the principal benefit to be derived from this science." He applies his theories to a problem proposed by Pappus but which had not been solved in general form.

Although our common system of coordinates is called *Cartesian* after Descartes, it should be noted that Descartes himself made no specification that the axes should be perpendicular to each other, and in fact he seldom if ever actually drew the y-axis.

The second book of the *Geometrie* discusses geometrical and mechanical curves, tangents, and normals. The third book is algebraic. Among its theorems are the following: "Every equation can have as many distinct roots as the number of dimensions of the unknown

* Preserved Smith, *History of Modern Culture*, 1930, Vol. I, p. 55.

† See David Eugene Smith and Marcia L. Latham, *The Geometry of René Descartes*, Chicago, 1925. This volume gives the first edition of the work in facsimile together with a translation and notes.

quantity in the equation." "An equation can have as many true roots as it contains changes of sign; and as many false roots as the number of times two + signs or two — signs are found in succession."*

It is interesting to note that although Descartes seems to have been the first to use x, y, and z for unknown quantities, he made occasional use of other notations. Professor Cajori suggests† that in correspondence he tended to use the notation familiar to the other parties. For instance, in a letter of 1640, he uses N for the first power of the unknown and C for its cube. At another time, a, b, and c represented known magnitudes and A, B, and C unknowns. He used exponents but customarily wrote aa instead of a^2. Descartes was familiar with works in which Recorde's sign of equality (=) was used, but, perhaps because Vieta and others used this sign to denote the arithmetic difference between two quantities, Descartes adopted the symbol ∞ for equality. This symbol became popular in Holland, it gained a following in France, and it even appeared in books printed in England. According to Cajori,‡ "the final victory of = over ∞ seems mainly due to the influence of Leibniz during the critical period at the close of the seventeenth century."

* Note that positive real numbers are *real* roots while negative roots are *false*.
† Florian Cajori, *History of Mathematical Notations*, 1928, Vol. I, p. 380.
‡ *loc. cit.* p. 306.

NEW BOOKS

The Teaching of Elementary Mathematics. By Charles Godfrey and A. W. Siddons. London, Cambridge University Press, 1931, 322 pages. Also published in New York by the Macmillan Company, price $2.50.

This book was planned a number of years ago by the late Professor Godfrey and Mr. Siddons who is now Senior Mathematical Master at Harrow School, England. The first part of the book which deals with the place of mathematics in education is the work of Professor Godfrey. The remainder of the book is the work of Mr. Siddons and covers the topics of elementary arithmetic, algebra, and geometry. The scope of the volume is given by Mr. Siddons in these words, "This book deals with the Teaching of Mathematics from about the age of 9 up to the School Certificate stage. It thus covers the mathematical work . . . up to the age of specialisation, except that it gives no detailed recommendations about Mechanics and Calculus."

The volume is divided into six parts each of which will be briefly discussed in the following paragraphs.

Part I of this treatise is especially full of thought-provoking statements. The views of Professor Godfrey are fundamentally modern and forwardlooking: sometimes they savor of American utilitarianism. He states, "But mainly it (arithmetic) is a complete collection of methods for solving all problems that have been set by all examiners since the invention of printing, a snowball that still grows, a burden to boyhood, a nightmare to mathematicians." "Algebra is perhaps the mathematical subject which gives the smallest return for the time spent on it. . . ." "They (teachers) want boys *to understand in order to manipulate correctly*, whereas their ideal should be reversed." "Now that geometry has become numerical and mathematical tables no longer a luxury, trigonometry cannot be kept out of the non-specialist course."

In Part II we find rather frank statements of teaching procedures which the author has found acceptable in his own practice. Although this section is interesting it is not as complete as might be desired by one seeking detailed advice on many pedagogical problems. In this section we find nineteen papers which show the type and difficulty of work that is expected from children of various ages. For a class of average age of 11 years and 10 months we find a good deal of algebra and geometry; $-3x = -21$ appears as an exercise in this group.

Part III treats the usual topics in arithmetic and it includes a short chapter on significant figures. Although the methods presented for some of the topics are more cumbersome than those in conventional use in the United States these methods are novel and should prove interesting from this point of view. The author finds place for a "Fifth Rule in Arithmetic. Common Sense." The contrast between the English system of money and our decimal system is very noticeable. It is suggested that a "metre rod divided into tenths" is useful in introducing decimals.

Part IV which deals with algebra contains some sound teaching suggestions of a general nature and then proceeds with several suggestions about each topic. The author prefers "to make

the equation and the problem the *raison d'etre* of the early work." His other main aims center about the power of generalization and the use of the formula and the idea of functionality.

Part V is perhaps the fullest and best section of the book. It deals with geometry from the early observational kind which the author thinks should be begun at the age of eleven and continues through a treatment of demonstrative geometry in two and three dimensions. The philosophy back of the treatment crops out in the statements, "The idea that young children can do things but cannot reason is responsible for much bad teaching." The author finds much value in *"assisted discovery"* and in the *"Heuristic method."* Some of the teaching points that are given should be helpful for beginning teachers.

The sixth part of the book is a very short treatment of trigonometry with a few notes on calculus and conics.

In general *The Teaching of Elementary Mathematics* is rich in suggestion and it is interesting to an American reader. While the treatment is not as well ordered as it is in some of our own books the style is informal and has a bit of charm. The teaching methods and techniques are not as detailed as one finds in many specialized books. The book should be welcomed in our country both for its intrinsic worth and for the information which it gives us about the teaching of mathematics in Great Britain. The reviewer was very pleased to have read the book and feels that he can recommend it for the libraries of teachers of mathematics and for reference use for institutions engaged in the training of teachers in this field.

BEN A. SUELTZ